QUARTERLY

OF

APPLIED MATHEMATICS

VOL. XIV JULY • 1956 NUMBER 2

Second-class mail privileges authorized at Providence, Rhode Island. Additional entry at Richmond, Virginia.

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

QUARTERLY OF APPLIED MATHEMATICS

Vol. XIV JULY, 1956 No. 2

AN EXPRESSION FOR GREEN'S FUNCTION FOR A PARTICULAR TRICOMI PROBLEM*

BY

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1. Introduction. This paper is concerned with the simplest equation of mixed type, known as the Tricomi equation

$$T(u) = zu_{xx} + u_{zz} = 0 \tag{1}$$

where u is the dependent variable, x and z the independent variables. Equation (1) is elliptic when $z > 0$, hyperbolic when $z < 0$. In this latter half plane the characteristics of (1) are the lines defined by $3x \pm 2(-z)^{3/2} =$ constant. The Tricomi problem consists of solving the equation $T(u) = 0$ in a domain Δ bounded by two concurrent characteristics AC and BC drawn in $z < 0$ and an arc AMB drawn in the half plane $z > 0$ when values of u are known along AMB and AC. Thus the mixed character of the equation is involved in the definition of the Tricomi problem and in fact the Tricomi problem is the typical problem for an equation of mixed type. The existence of a solution for such a problem has been proved by F. Tricomi [1]** himself in his fundamental paper. A quite different type of proof has been given by P. Germain and R. Bader [2], [3]. They have considered in particular the special case for which AMB is a "normal" curve, according to Tricomi's terminology—that is to say an arc defined by $(x - x_0)^2 + y^2 = R^2$, $3y = 2z^{3/2}$, with x_0 and R given constants and have shown that in such a case it is possible to give an explicit solution of the Tricomi problem. Such a Tricomi problem will be called a "normal" Tricomi problem. This result was quite interesting from a theoretical point of view and permitted a very simple proof of the existence of the solution of the Tricomi problem to be given. More recently [4], [5], and [6] it was shown that such a "normal" problem was of particular interest in the application of an approximate method to subsonic and transonic flows involving jets or wedges. However, the solution given previously was not found quite satisfactory from the computational point of view. It was emphasized, [2], that a transformation involving three parameters (transformation related to the Poincare's geometry) can be associated with the Tricomi equation. As a result, the solution of a "normal" problem can be simply derived from the special case in which Δ is the region Δ_0 defined by $x > 0$ for $z > 0$, $3x > 2(-z)^{3/2}$ for $z < 0$. The values of u along oz $(x > 0)$ are known; u is also given either along AC, or along the

*Received May 5, 1955. The results presented in this paper were obtained in the course of research sponsored by the National Advisory Committee for Aeronautics under Contract NAW-6323 while the author was Visiting Professor of Applied Mathematics at Brown University.

**Numbers in brackets refer to the bibliography at the end of the paper.

characteristic at infinity; in the former case, which will be called the direct problem, the solution must be regular at infinity [2]; in the latter case it will be called the conjugate problem. An inversion with center at the origin allows one to reduce one of these problems to the other. A few comments are needed on the definition of the Green's function of a Tricomi problem. This notion was introduced in [2], [3] for the case of a Tricomi equation and generalized in [7], [8] for a wider class of equations of mixed type. It arises when one looks for the possibility of writing the expression for the solution of a Tricomi problem as a linear functional of the data. It can be shown that to every point P inside Δ, one can make correspond a function $g_P(M)$, called the Green's function of the Tricomi problem for the given domain Δ. This function is continuous for M inside Δ, (if some singular lines or points are excluded), and has the following fundamental properties: Given one solution u of (1) in Δ, it is possible to find by application of the Green's formula the value of $u(P)$ in terms of some integrals involving only (besides the value of g_P and its derivatives), the value of u along AMB and AC. Moreover, g_P as function of the coordinate of M is a fundamental solution of (1) and g_P is zero on AMB and BC; g_P as a function of the coordinates of P, M being kept fixed inside Δ, is a fundamental solution of (1) and takes the value zero when P is on AMB or on AC. In other words, as a function of M, $g_P(M)$ satisfies some boundary conditions for the conjugate problem; as a function of P, it satisfies some boundary conditions for the direct problem. The notion of fundamental solution* for an equation of mixed type is also discussed in [7], [8]. When the point P is in the elliptic half plane ($z > 0$), g_P as a function of M is regular everywhere in the open domain Δ, except in the neighborhood of P, near which it has the classical logarithmic singularity. When the point P is in the hyperbolic half plane

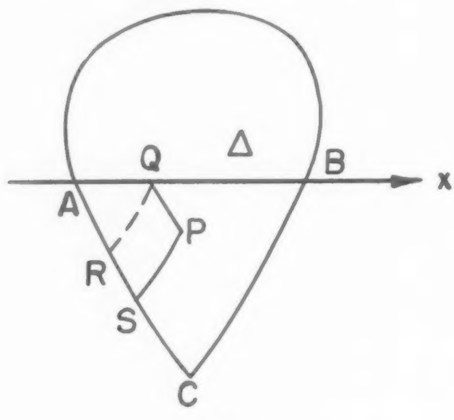

Fig. 1.

($z < 0$), the Green's function as function of M is singular on the characteristics shown in Fig. 1: it has some discontinuities along PQ and PS, proportional to the values of the Riemann function along these lines, and it becomes logarithmically infinite in the neighborhood of the reflected characteristic QR. A similar behavior is of course valid for g as a function of P when M is fixed inside Δ.

*In the usual French terminology such a solution is called "elementary".

The following developments are the result of two remarks. First it was shown in [7] and [8] how it is possible to build the Green's function of a strip for a class of differential equations, even if the strip lies in a mixed region. Second, for the Tricomi equation (1) new independent variables can be introduced in such a way that the Green's function of the Tricomi problem corresponding to Fig. 2 is transformed into the Green's function

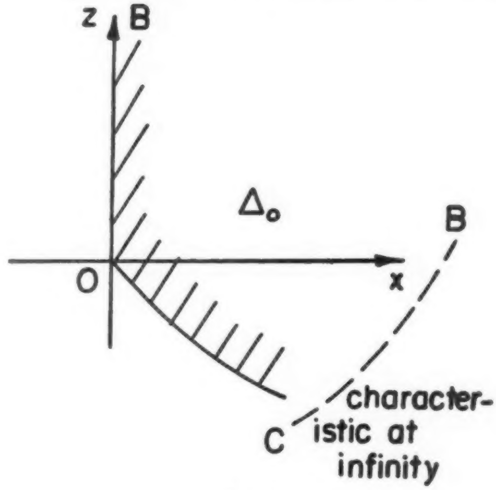

FIG. 2.

of a strip in a mixed domain. Thus it is possible to apply a previous technique with some minor differences in order to obtain the required result.

2. Transformations of the fundamental equation. The following definitions were introduced in [2]:

$$3y = 2z^{3/2}, \qquad r^2 = x^2 + y^2, \qquad x = rt. \tag{2}$$

In Δ_0, r^2 as defined by (2) is positive (zero on the characteristic OC), and accordingly r will be assumed real and positive. With r and t as new independent variables, the operator $T(u)$ becomes:

$$T(u) = zL(u) = z\{u_{rr} + (1 - t^2)r^{-2}u_{tt} + (4/3)(r^{-1}u_r - r^{-2}tu_t)\}. \tag{3}$$

Now it is convenient to introduce the new variable ξ defined by $r = \exp \xi$, and with ξ and t as independent variables (3) becomes

$$T(u) = zL(u) = r^{-2}zM(u) = r^{-2}z\{(1 - t^2)u_{tt} + u_{\xi\xi} + 1/3(u_\xi - 4tu_t)\}. \tag{4}$$

The domain Δ_0 is mapped into the half plane $t > 0$ of the ξ, t plane. For $0 \leq t \leq 1$, the equation $M(u) = 0$ is of the elliptic type, and for $t > 1$, hyperbolic. Another form can be used with ξ and λ as independent variables where

$$\lambda = \int_t^1 (1 - v^2)^{-2/3}\, dv,$$

namely

$$\begin{aligned} T(u) &= r^{-2}zM(u) = r^{-2}z(1 - t^2)^{-1/3}N(u) \\ &= r^{-2}z(1 - t^2)^{-1/3}\{u_{\lambda\lambda} + (1 - t^2)^{1/3}(u_{\xi\xi} + (1/3)u_\xi)\} \end{aligned} \tag{5}$$

The Green's function we are looking for is a fundamental solution of (1); precisely, such a fundamental solution e_P is a solution of

$$T(u) = \delta_{x_0, z_0}. \tag{6}$$

where δ_{x_0, z_0} is the Dirac distribution [9] at the point $P(x_0, z_0)$. An equivalent definition is, that for every φ which is continuously twice differentiable and zero outside some compact subset of Δ_0,

$$\varphi(x_0, z_0) = \iint e_P T(\varphi)\, dx\, dz. \tag{7}$$

Thus, if we express e_P with the variables ξ and t (ξ_0 and t_0 are the values of these variables for the point P), (7) can be written

$$\varphi(\xi_0, t_0) = -\iint e_P z r^{-2} M(u)(3/2) r^3 z^{-2}\, d\xi\, dt$$

and e_P is a solution of

$$M(u) = -(2/3) r_0^{-1} z_0 \delta_{\xi_0, t_0}; \tag{8}$$

similarly, with ξ and λ, e_P is found to satisfy

$$N(u) = (2/3)^{1/3} r_0^{-1/3} \delta_{\xi_0, \lambda_0}. \tag{9}$$

δ_{ξ_0, t_0} and $\delta_{\xi_0, \lambda_0}$ are the Dirac distributions for the two variables ξ, t, and ξ, λ respectively.

Although the equation $N(u) = 0$ does not belong, strictly speaking, to the class considered in [7], and [8], it can be studied by the same method. In order to show the extension of this method, a simple Dirichlet problem will be considered in the next section.

3. Singular Dirichlet problem for the region $x > 0$, $z > 0$.

A new expression of the Green's function will now be derived by the same method which will be used later for the Tricomi problem.

In the ξ, t plane this region is mapped into the strip $0 < t < 1$; in the ξ, λ plane into a similar strip $0 < \lambda < \lambda_1$, (λ_1 being the value of λ which corresponds to $t = 0$). We introduce the Fourier transform $U(\alpha, \lambda)$ of $u(\xi, \lambda)$, $U = \mathfrak{F}u$, which for summable functions may be written

$$U = \mathfrak{F}u = \int_{-\infty}^{+\infty} \exp(-2i\pi\alpha\xi)\, u d\xi, \qquad u = \mathfrak{F}^{-1}U = \int_{-\infty}^{+\infty} \exp(2i\pi\alpha\xi)\, U d\alpha \tag{10}$$

and use, as in [7], the extension of this transform to distributions [9]. It is easy to show [7] that the transform of (9) is

$$n(U) = U_{\lambda\lambda} + (1 - t^2)^{1/3}[(2/3)i\pi\alpha - 4\pi^2\alpha^2]U = (2/3)^{1/3} r_0^{-1/3} \exp(2i\pi\alpha\xi_0)\, \delta_{\lambda_0} \tag{11}$$

where δ_{λ_0} is the Dirac distribution of one variable λ at the point $\lambda = \lambda_0$. In order to solve (11), the following notation is introduced: $S_1(\lambda, \alpha)$ and $S(\lambda, \alpha)$ are the solutions of $n(U) = 0$ which satisfy

$$S_1(\lambda_1, \alpha) = 0, \qquad S(0, \alpha) = 0, \qquad \frac{\partial}{\partial\lambda} S_1(\lambda_1, \alpha) = 1, \qquad \frac{\partial}{\partial\lambda} S(0, \alpha) = 1.$$

The Fourier transform E_P of e_P is then defined by $E_P = (3/2)^{-1/3} r_0^{-1/3} \exp(-2i\pi\alpha\xi_0) E_P^*$

$$E_P^* = \begin{cases} h S_1(\lambda, \alpha) S(\lambda_0, \alpha), & \lambda_0 < \lambda < \lambda_1, \\ h S_1(\lambda_0, \alpha) S(\lambda, \alpha), & 0 < \lambda < \lambda_0, \end{cases} \tag{12}$$

h being such that the jump in the value of the first derivative of E_P^* with respect to λ at $\lambda = \lambda_0$ be equal to $+1$. Obviously $h^{-1} = S(\lambda_1, \alpha) = -S_1(0, \alpha)$.

On the other hand S_1 and S are easily found from the first Darboux solutions; set

$$s = 2i\pi\alpha + 1/6, \qquad \tau = t^2, \tag{13}$$

one can write

$$\left.\begin{aligned} S_1(\lambda, \alpha) &= -tF(7/12 + s/2, 7/12 - s/2, 3/2, \tau), \\ S(\lambda, \alpha) &= 3/2(1 - \tau)^{1/3}F(5/12 + s/2, 5/12 - s/2, 4/3, 1 - t), \end{aligned}\right\} \tag{14}$$

where $F(a, b, c, \tau)$ denotes the usual hypergeometric function. Accordingly h can be expressed in terms of gamma functions

$$h = 2\Gamma(11/12 + s/2)\Gamma(11/12 - s/2)[\Gamma(1/2)\Gamma(1/3)]^{-1}. \tag{15}$$

Now we must investigate the most convenient form in which to write the inverse transform. The equation (11) will be identical to equation (20) of [7] (apart from some obvious changes in the notation) if we introduce the variable β defined by

$$4\pi^2\beta^2 = 4\pi^2\alpha^2 - (2/3)i\pi\alpha \tag{16}$$

and choose the branch which reduces for large values of $|\alpha|$ to $\beta \cong \alpha - i/12\pi$. Thus the asymptotic behavior* of the solutions of (11) for large values of $|\beta|$ are given by some formulae similar to formulae** (23) and (37) of [7]. The real axis of the α plane and the corresponding line $Rl\{s\} = 1/6$ of the s plane are mapped into the contour C in the β plane (Fig. 3). Therefore that the right hand side of (12) is a distribution whose

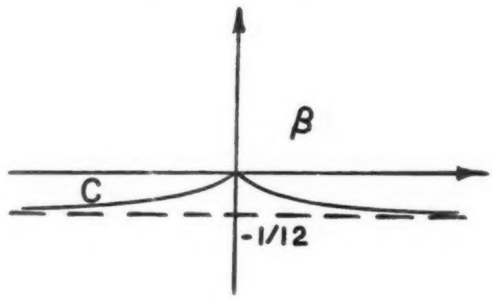

FIG. 3.

inverse transform can be written using an integral in the s plane and therefore the value of e_P is given by

$$e_P = (2/3)^{1/3}(2i\pi r_0^{1/3})^{-1}\int_{1/6-i\infty}^{1/6+i\infty} E_P^*(rr_0^{-1})^{s-1/6}\, ds \tag{17}$$

because $r = \exp \xi$. The reader will recognize in this formula the Mellin transform which in fact may be applied to the operator $L(u)$. From such a solution, the expression for

*The general study of the asymptotic behavior of these solutions is found in [10].

**Or the formulae (9) and (10) of [8].

the doublet at the point $M_1(0, z_1)$ may be derived easily. The Green's formula for equation (1) shows that a solution u defined in $x > 0, z > 0$ which is zero along Ox, is given by

$$u(P) = -\int_0^\infty zu(0, z)(e_P)_x \, dz.$$

This proves that the doublet $d_{M_1}(P)$ in $M_1(0, z_1)$ is

$$d_{M_1}(P) = -z_1 \frac{\partial}{\partial x} e_P(0, z_1, x_0, z_0). \tag{18}$$

It is possible to evaluate the right hand side of (17) and (18) by series expansions. For instance, if $\mathfrak{P}_p(\tau)$ is the polynomial of degree p

$$\mathfrak{P}_p(\tau) = F(-p, 11/6 + p, 3/2, \tau),$$

then

$$d_{M_1}(P) = (2/3)^{1/3} z_1 t_0 (1 - \tau_0)^{1/3} \sum_0^\infty 4\Gamma(3/2 + p)\Gamma(11/6 + p)[\pi\Gamma(4/3 + p)p!]^{-1}$$

$$\times \mathfrak{P}_p(\tau_0) \begin{cases} r_1^{-4/3}(r_1 r_0^{-1})^{2p+2} & \text{if} \quad r_1 < r_0 , \\ r_0^{-4/3}(r_0 r_1^{-1})^{2p+3} & \text{if} \quad r_1 > r_0 . \end{cases}$$

In practical applications to boundary value problems the quantity $\partial/\partial x_0 \, d_{M_1}(0, z_0)$ is very important. The following result is easily obtained

$$\partial/\partial x_0 \, d_{M_1}(0, z_0) = \begin{cases} (3/2)^{1/3} 4\Gamma(11/6)\Gamma(3/2)[\pi\Gamma(4/3)]^{-1} r_1^{4/3} r_0^{-3} F(11/6, 3/2, 4/3, r_1^2 r_0^{-2}), \\ \qquad \text{if} \quad r_1 < r_0 , \\ (3/2)^{1/3} 4\Gamma(11/6)\Gamma(3/2)[\pi\Gamma(4/3)]^{-1} r_1^{-7/3} r_0^{2/3} F(11/6, 3/2, 4/3, r_0^2 r_1^{-2}), \\ \qquad \text{if} \quad r_1 > r_0 . \end{cases} \tag{19}$$

These results can be checked with those obtained by different methods (methods of discontinuous integrals or transforms [11], [12], and [13], or the direct method given in [2] and [3].

4. Green's function for the Tricomi problem. The same method can be applied in order to find an expression for the Tricomi problem of Fig. 2. The domain to be considered in the ξ, t plane is the half plane $t > 0$ which corresponds to the strip $\lambda_1 > \lambda > \lambda_2$ (λ_2 negative) in the ξ, λ plane. One must again form a solution of (11) with convenient boundary conditions. As previously, the solution S_1 (λ, α)—see (14)—is needed. The second solution we need in order to define E_P as in (12) is defined by a property of asymptotic behavior. According to the general theory, [7, 8] we have two possibilities depending on the orientation of the solutions. It was shown that these two solutions, which correspond to the two Tricomi problems which can be defined in the domain Δ_0, are 1) a function $H_1(\lambda, \alpha)$, a solution of $n(u) = 0$ which is real for $\beta = i\beta'$ (β' positive), see (16), and which tends uniformly toward zero for any $\lambda_2 < \lambda < \lambda_1$ when β' tends towards $+\infty$; 2) a function $H_2(\lambda, \alpha)$ which is simply defined by $H_2(\lambda, \beta) = H_1(\lambda, \beta e^{-i\pi})$. Let us note that (13), and (16) give $-4\pi^2\beta^2 = 4\pi^2\beta'^2 = s^2 - 1/36$ and $s \sim -2\pi\beta'$ for β' sufficiently large. Noting also that $H_1(\lambda, \alpha)$ for $\beta = i\beta'$ must tend towards zero

when $\lambda \to \lambda_2$, that is to say $t \to +\infty$, it is evident that H_1 and H_2 are given by the following expressions, solutions of $n(U) = 0$ (see [2] p. 11)

$$\begin{aligned} H_1 &= t(\tau - 1)^{-7/12+s/2}F(7/12 - s/2, 11/12 - s/2, 1 - s, [1 - \tau]^{-1}), \\ H_2 &= t(\tau - 1)^{-7/12-s/2}F(7/12 + s/2, 11/12 + s/2, 1 + s, [1 - \tau]^{-1}). \end{aligned} \tag{20}$$

Therefore, if $G_P^{(1)}$ or $G_P^{(2)}$ are the Fourier transforms of $g_P^{(1)}$ and $g_P^{(2)}$—these are the two functions we are looking for—we may write as in (12),

$$G_P^{(j)} = (3/2)^{-1/3}r_0^{-1/3} \exp(-2i\pi\alpha\xi_0)G_j^* \qquad (j = s, 2)$$

with

$$G_j^* = \begin{cases} h_jS_1(\lambda, \alpha)H_j(\alpha_0, \alpha), & \lambda_0 < \lambda < \lambda_1, \\ h_jS_1(\lambda_0, \alpha)H_j(\lambda_0, \alpha), & \lambda_2 < \lambda < \lambda_0, \end{cases} \tag{21}$$

h_j being such that the jump of the first derivative of G_j^* with respect to λ for $\lambda = \lambda_0$ is equal to $+1$. Thus h_j is the inverse of the Wronskian of S_1 and H_j, then

$$h_j = 3^{1/2}\Gamma(11/12 - js/2)\Gamma(7/12 - js/2)[2\Gamma(1/2)\Gamma(1 - js)\sin\pi(1/4 - js/2)]^{-1}. \tag{22}$$

Now, as in (17), $g_P^{(j)}$ is given by the Mellin transform

$$g_P^{(j)} = (2/3)^{1/3}(2i\pi r_0^{1/3})^{-1}\int_{1/6-i\infty}^{1/6+i\infty} G_j^*(rr_0^{-1})^{s-1/6}\,ds. \tag{23}$$

It is not difficult, of course, to verify for this result some properties of the Green's function which have been proved previously [2]. For instance, the "symmetry" property $g_P^{(1)}(M) = g_M^{(2)}(P)$, and the "inversion" property

$$g_P^{(1)}(M) = g_P^{(1)}(r, t; r_0, t_0) = (r_0r^{-1})^{1/3}g_P^{(2)}(r_0^2r^{-1}, t; r_0, t_0).$$

For practical purposes, the most important thing is to find the expression for the doublet $D_{M_1}^{(j)}(P)$ at the point M_1 $(0, z_1)$ the doublet associated with the Tricomi problems in Δ_0. The same argument used previously in order to derive (18) shows that

$$D_{M_1}^{(j)}(P) = -z_1\frac{\partial}{\partial x}g_P^{(j)}(0, z_1, x_0, z_0) \tag{24}$$

and thus

$$D_{M_1}^{(j)}(P) = \left(\frac{2}{3}\right)^{1/3}\frac{z_1}{r_1r_0^{1/3}}\frac{1}{2i\pi}\int_{1/6-i\infty}^{1/6+i\infty} h_jH_j(\lambda_0, \alpha)\left(\frac{r_1}{r_0}\right)^{s-1/6}ds. \tag{25}$$

In many applications, especially in the transonic problems we have in mind, the only thing which is important to know is the value of the normal derivative along Oz, a value which can be computed with an integral if we know the value of $\partial/\partial x_0\, D_{M_1}^{(j)}(0, z_0) = K_j(z_1, z_0)$. This one can be expressed as some kind of generalized hypergeometric function. In fact

$$K_1(z_1, z_0) = -(3/2)^{1/3}4r_1^{-1/2}r_0^{-7/6}G_{4,4}^{3,1}\left[\left(\frac{r_1}{r_0}\right)^2 \middle| \begin{matrix} 1/4, 1/12, & 5/12, 3/4 \\ 1/4, 7/12, & 11/12, 3/4 \end{matrix}\right], \tag{26}$$

$G^{m;n}_{p;q}$ being a Meijer's G-function, (see [14], p. 206). From a straightforward application of the formulae given in [14] p. 208, one obtains with the classical notation ${}_pF_q(a_1 \cdots, a_p\,; b_1 \cdots, b_q\,; z)$ for the generalized hypergeometric functions.

First case: $r_0 < r_1$,

$$K_1(z_1\,, z_0) = -(3/2)^{1/3}16r_0^{1/3}[3^{3/2}\pi r_1^2]^{-1}\,{}_3F_2(4/3,\ 5/3,\ 1;\ 5/6,\ 7/6,\ r_0^2r_1^{-2}). \tag{27}$$

Second case: $r_1 < r_0$,

$$\begin{aligned} K_1(z_1\,, z_0) = &-(3/2)^{1/3}2[3^{3/2}\pi r_0^{5/3}]^{-1}\,{}_3F_2(7/6,\ 5/6,\ 1;\ 1/3,\ 2/3;\ r_1^2r_0^{-2}) \\ &-(2/3)^{2/3}r_1^{2/3}\Gamma(1/2)\Gamma(7/6)[\pi r_0^{7/3}\Gamma(2/3)]^{-1}\,{}_2F_1(3/2,\ 7/6;\ 2/3;\ r_1^2r_0^{-2}) \\ &-(3/2)^{1/3}5r_1^{4/3}[3\pi r_0^3]^{-1}\,{}_2F_1(11/6,\ 3/2;\ 4/3;\ r_1^2r_0^{-2}). \end{aligned} \tag{28}$$

These formulae seem to be convenient for numerical computations.

The values of $K_2(z_1\,, z_0)$ may be derived from (27) and (28) with

$$z_0K_1(z_1\,, z_0) = z_1K_2(z_0\,, z_1). \tag{29}$$

Let us conclude this section by recalling that $g_P^{(1)}$ is the Green's function for the direct Tricomi problem for P in the hyperbolic half plane (Fig. 4), this function has its

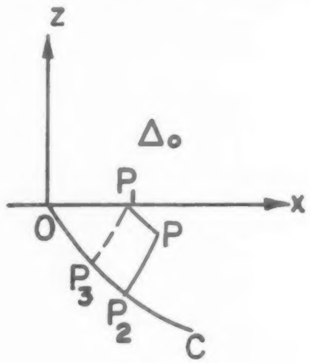

FIG. 4

singularities along PP_1, PP_2 (discontinuities proportional to $(-z)^{-1/4}$) and its logarithmic singularities along P_1P_3. Similarly, $g_P^{(2)}$ is the Green's function of the conjugate Tricomi problem in Δ_0 and has its singularities along PP_1, PP_2, PP_3 (Fig. 5). The doublet $D_M^{(1)}(P)$ is the function which allows one to solve the direct Tricomi problem where the value of the solution along OC is zero, $D_M^{(2)}(P)$ the doublet which allows one to solve the conjugate Tricomi problem, when the required solution is zero on the characteristic at infinity. When r is infinitely small the limiting value of $D_M^{(2)}(P)$ (r_0 finite) must be proportional (the factor may depend upon r_1) to the first Darboux solution which gives the asymptotic behavior at infinity of the flow around the profile of the Mach number 1. This principal value of $D_M^{(2)}(P)$ is obtained by considering the positive residue in (25). It is proportional to

$$\begin{aligned} r_0^{-5/3}t_0(t_0^2 - 1)^{-4/3}F(4/3,\ 5/3,\ 5/2,\ [1 - t_0^2]^{-1}) \\ = Kr_0^{-3}\{(r - x)^{1/3}(3x - r) - (r + x)^{1/3}(3x + r)\}, \end{aligned} \tag{30}$$

where K is a numerical constant.

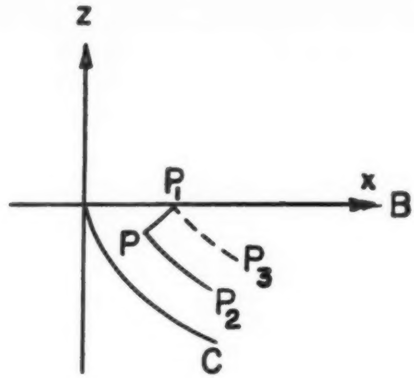

FIG. 5.

5. A special generalized Tricomi problem. It was often suggested, for instance in [15], that a generalized Tricomi problem could be considered, for which the solution is to be found in a domain Δ, bounded by an arc AB drawn in the elliptic half plane with end points A and B on the x axis, an arc AC in the hyperbolic half plane with a time-like direction if the direction AB is taken as the time direction, and an arc BC of a characteristic. The data for such a problem are the values of the solution u along the arc AB and the arc AC. By a transformation of the group of the Poincare geometry associated with the Tricomi equation, it is always possible to reduce this problem to a similar one in which BC is the characteristic at infinity. Now consider the special case for which the arcs OB and OC are such that t is constant along each of these arcs. The corresponding domain in the ξ, λ plane is mapped onto a strip $\lambda_4 < \lambda_3$ parallel to the ξ axis. Thus, it is clear that the previous method must allow one to give the solution of this generalized Tricomi problem.

In the ξ, λ plane we have to start with the fundamental solution $e_P^*(M)$ of $N(u) = 0$, which is "orientated" in the direction opposite to that of the ξ axis. $P(\xi_0, \lambda_0)$ is the singular point of $e_P^*(M)$ and the function has a singular behavior along the singular characteristics schematically represented on (Fig. 6); $e_P^*(M)$ has a discontinuity on each solid characteristic and a logarithmic singularity along each dotted characteristic. On the other hand $e_P^*(M)$ will be chosen in such a way that it vanishes when M lies on $\lambda = \lambda_3$ or $\lambda = \lambda_4$. The expression of such a solution was given in [7]. Let us call $S_3(\lambda, \alpha)$ and $S_4(\lambda, \alpha)$ the solutions of $n(U) = 0$ which satisfy the following conditions:

$$S_3(\lambda_3, \alpha) = 0, \quad S_4(\lambda_4, \alpha) = 0, \quad \partial/\partial\lambda\ S_3(\lambda_3, 0) = 1, \quad \partial/\partial\lambda\ S_4(\lambda_4, 0) = 1. \tag{31}$$

Using this notation, $e_P^*(M)$ is the inverse Fourier transform (ξ_0 is taken equal to 0 in (32)) of

$$E_P^* = \begin{cases} WS_3(\lambda, \alpha)S_4(\lambda_0, \alpha), & \lambda_0 < \lambda < \lambda_3, \\ WS_4(\lambda, \alpha)S_3(\lambda_0, \alpha), & \lambda_4 < \lambda < \lambda_0, \end{cases} \tag{32}$$

with*

$$W^{-1} = S_3(\lambda_4, \alpha) = -S_4(\lambda_3, \alpha). \tag{33}$$

*The functions $S_3(\lambda, \alpha)$, $S_4(\lambda, \alpha)$ may be easily expressed using the hypergeometric functions of the first Darboux solutions.

It was shown that expressions (32) are meromophic functions with respect to the variable β defined by (16), which have poles for a sequence of real values of β and for a sequence of purely imaginary values of β and that, in order to obtain the fundamental solution with the orientation given by Fig. 6, the integral which gives the inverse Fourier trans-

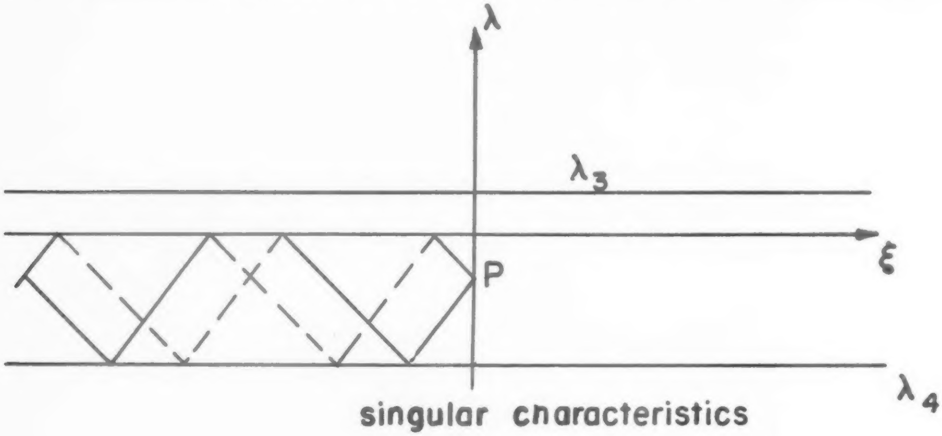

FIG. 6.

form of (32) must be taken in the β plane on a line which leaves the real poles below and the imaginary poles with positive ordinates above. Recalling the relations which join s and β: $s^2 = 1/6 - 4\pi^2\beta^2$, $s \sim 2i\pi\beta$ for $|\beta|$ large, it is clear that:

$$e_P^*(M) = (2i\pi)^{-1} \int_{\mathcal{C}_1} E_P^*(rr_0^{-1})^{s-1/6}\, ds, \tag{34}$$

where $\mathcal{C}_1$ is schematized in the figure 7. It was shown also that $e_P^*(M)$ tends towards zero when ξ tends towards $+\infty$ for every $\lambda_4 < \lambda < \lambda_3$.

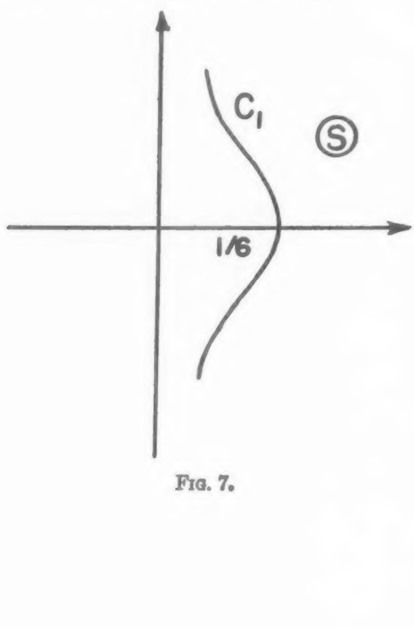

FIG. 7.

Now, according to (9) the Green's function $g_P(M)$ for the special problem we consider is

$$g_P(M) = (2/3)^{1/3} r_0^{-1/3} e_P^*(M). \tag{35}$$

This function is zero on the characteristic at infinity, on the arc OB and on the arc OC, but has a singular point in 0. When P lies in the elliptic part of Δ_0, $g_P(M)$ has the classical logarithmic singularity in the vicinity of P, when P lies in the hyperbolic part, $g_P(M)$ has a singular behavior along the characteristic lines schematized in Fig. 8. Although

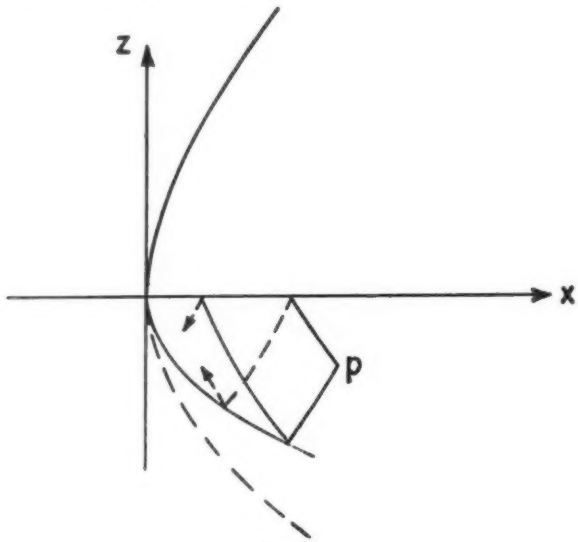

FIG. 8.

we do not want to enter into all the details, it is quite clear that the previous result allows us to state the uniqueness and existence theorems for the solution of this special generalized Tricomi problem when the value of the unknown function is prescribed along $OB(t = t_3)$ and $OC(t = t_4)$, by application of the Green's formula. A small difficulty arises in the vicinity of o because of the singularity of the Green's function near the origin, since the application of the Green's formula gives rise to a series of integrals. But this series can be shown to be convergent. Such a situation always arises in classical problems for hyperbolic equations in connection with what E. Picard [16] has called a fourth boundary value problem, (see for instance [17]).

6. Concluding remarks. The previous results allow one to simplify a great deal the proof of the existence of the solution of the Tricomi problem. To outline very briefly such a proof, it is first of all clear that the result of Section 4 gives the direct solution of the problem when the contour AMB (Fig. 1) is a normal contour. It is also possible to give a direct solution when the contour AMB is, in the x, y plane, any circular arc with end-points A and B, by using the result of Section 5. Now the general case when the contour AMB is arbitrary can be solved by using the Schwartz process as in [2] and the sequence of solutions obtained by this method converges towards the solution by application of the maximum principle [2].

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DETERMINATION OF COEFFICIENTS OF CAPACITANCE OF REGIONS BOUNDED BY COLLINEAR SLITS AND OF RELATED REGIONS*

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1. Introduction. In the study of electrostatic field problems the principal objective usually is to determine the potential and its gradient (the field strength) throughout a given domain bounded by a system of conductors. Frequently, however, it is necessary only to determine certain constants of capacitance. In this paper we consider the latter problem for a certain class of plane domains.

Let D be a domain consisting of the entire plane with any finite number of slits along a single line. Several formulas for the coefficients of capacitance of such a domain are derived, two of which appear to be well suited for numerical computations. One of these formulas is based on the explicit representation of the potential as the real part of an analytic function [1] while the other formula has the feature of requiring a knowledge of the potential only on the line containing the boundary components of D; it does not involve any derivatives of the potential. A convenient method for determining the field along the line containing the boundary components has been given by the author in a previous paper [2].

Since the coefficients of capacitance of a domain are invariant under conformal mapping, the formulas which are derived may be employed to compute the coefficients of any domain which can be conformally mapped upon a domain D of the type described above. This procedure can be applied to a certain class of domains which are of practical interest. In these cases the mapping problem involves essentially only simply-connected domains rather than multiply-connected domains. One particular case of interest, the 'bi-filar shielded cable' is considered in some detail, and as an illustration of the procedure, the coefficients of capacitance are evaluated numerically for one such domain.

2. Regions bounded by collinear slits. We consider here the problem of determining effectively the so-called coefficients of capacitance of a domain D consisting of the extended (x, y)-plane with a finite number, m, of collinear slits, cut along what may be assumed to be the x-axis.

We recall the definition of these coefficients:

$$p_{jk} = -\oint_{C_k} \frac{\partial u_j}{\partial n}\, ds, \qquad (j, k = 1, 2, \cdots, m), \tag{2.1}$$

where u_j, the harmonic measure of the jth boundary component, is the harmonic function whose boundary values are unity on the jth component and zero on all other components (the existence and uniqueness of this harmonic function is well known from the theory of the Dirichlet problem); C_k is any curve, described in the positive sense, surrounding only the kth component; and $\partial/\partial n$ indicates differentiation in the direction of the outward normal. As follows immediately from the Cauchy-Riemann equations, the p_{jk} may be

*Received May 17, 1955. The research reported in this article was done at the Institute of Mathematical Sciences, New York University, and was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

defined alternatively as the increment in the harmonic function v_j conjugate to u_j which results when C_k is described once in the *negative* sense.

We also recall several important properties of these coefficients of capacitance:

$$p_{jk} = p_{kj} ; \tag{2.2a}$$

$$\sum_{k=1}^{m} p_{jk} = 0, \qquad j = 1, 2, \cdots, m; \tag{2.2b}$$

$$p_{jj} > 0; \tag{2.2c}$$

$$p_{jk} < 0, \qquad j \neq k. \tag{2.2d}$$

The p_{jk} are, of course, conformal invariants. Hence we may assume that D consists of the entire plane minus a finite number of slits lying on the x-axis, one of which extends to infinity in both directions (see Fig. 1); this configuration can always be realized by

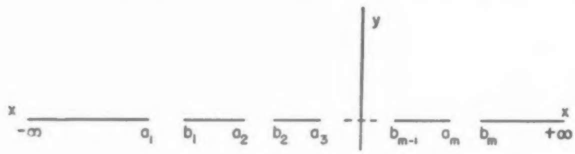

FIG. 1.

a suitable inversion. It will be seen that this assumption eliminates the possibility of any convergence difficulties. For brevity such domains will be called slit-domains. We number the finite slits $1, 2, \cdots, m - 1$ from left to right; and the infinite slit is the mth.

We proceed to derive various formulas for the quantities p_{jk} which may prove useful for numerical computations. First we shall employ the definition (2.1); later we shall derive a formula based on the alternative definition given following (2.1).

Taking into account (2.2a) and (2.2b) we see that it suffices to determine the p_{jk} with $j < m$, $k < m$. Let $f_j(x) = u_j(x, 0)$, so that, in particular, $f_j(x) = 0$ for $x \leq a_1$, $x \geq b_m$. By the Poisson formula we have, for any point off the x-axis:

$$u_j(x, y) = \frac{|y|}{\pi} \int_{a_1}^{b_m} \frac{f_j(\xi)\, d\xi}{(\xi - x)^2 + y^2}. \tag{2.3}$$

We take as the curve C_k a rectangle with vertical sides passing through the gaps (a_k, b_k) and (a_{k+1}, b_{k+1}). It is easily found from (2.3), by differentiating under the integral sign, that for any value of x satisfying $a_1 \leq x \leq b_m$

$$\left| \frac{\partial u_j}{\partial x} \right| \leq \frac{2A^2}{\pi |y|^3}, \qquad \left| \frac{\partial u_j}{\partial y} \right| \leq \frac{2A}{\pi y^2}, \qquad A = \max(|a_1|, |b_m|) \tag{2.4}$$

(one uses, of course, the fact that $|f_j(\xi)| \leq 1$). Hence as the horizontal sides recede to infinity the contribution to the integral (2.1) from these sides vanishes, and therefore (2.1) may be written as follows:

$$p_{jk} = I_{j,k} - I_{j,k+1}, \tag{2.5}$$

where

$$I_{j,r} = \int_{-\infty}^{\infty} \frac{\partial u_j}{\partial x}\, dy, \qquad a_r < x < b_r \tag{2.6}$$

(of course $p_{jm} = I_{j,m} - I_{j,1}$).

We wish to rewrite $I_{i,r}$ in such a form as to involve only the values of u_i on the x-axis, i.e., the values $f_i(x)$. If one expresses $(\partial u_i)/\partial x$ by differentiating (2.3), inserts this expression into (2.6), and interchanges the order of integration, one obtains the following expression for $I_{i,r}$:

$$I_{i,r} = \frac{2}{\pi} P \int_{a_1}^{b_m} \frac{f_i(\xi)\, d\xi}{\xi - x}, \qquad a_r < x < b_r . \tag{2.7}$$

Here the symbol P denotes the Cauchy principal value of the integral.

A second expression for $I_{i,r}$ is obtained by the following artifice. Since the right side of (2.6) gives the same value for any choice of x in the rth gap, we integrate both sides of (2.6) over this gap. Then we obtain

$$(b_r - a_r)I_{i,} = \int_{a_r}^{b_r} \int_{-\infty}^{\infty} \frac{\partial u_i}{\partial x}\, dy\, dx.$$

Interchanging the order of integration [this is easily justified with the aid of the first inequality in (2.4)] we obtain

$$\begin{aligned}(b_r - a_r)I_{i,r} &= \int_{-\infty}^{\infty} \int_{a_r}^{b_r} \frac{\partial u_i}{\partial x}\, dx\, dy = \int_{-\infty}^{\infty} \{u(b_r , y) - u(a_r , y)\}\, dy \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} | y | \left\{ \int_{a_1}^{b_m} f_i(\xi) \left[\frac{1}{(\xi - b_r)^2 + y^2} - \frac{1}{(\xi - a_r)^2 + y^2} \right] d\xi \right\} dy.\end{aligned} \tag{2.8}$$

Now we interchange the order of integration once more, and obtain the following expression for $I_{i,r}$:

$$I_{i,r} = \frac{2}{\pi(b_r - a_r)} \int_{a_1}^{b_m} f_i(\xi) \ln \left| \frac{\xi - a_r}{\xi - b_r} \right| d\xi. \tag{2.9}$$

A third expression for $I_{i,r}$ is obtained as follows. Since $f_i(\xi)$ is continuous and $f_i'(\xi)$ and $\ln | (\xi - a_r)/(\xi - b_r) |$ are absolutely integrable, an integration by parts enables us to rewrite (2.9) in the form:

$$I_{i,r} = \frac{-2}{\pi(b_r - a_r)} \int_{a_1}^{b_m} f_i'(\xi)[(\xi - a_r) \ln | \xi - a_r | - (\xi - b_r) \ln | \xi - b_r |]\, d\xi; \tag{2.10}$$

the integrated term vanishes since $f_i(a_1) = f_i(b_m) = 0$.

Equations (2.7), (2.9), (2.10), together with (2.5), give three formulas for the coefficients p_{ik} which employ only the values of u_i on the x-axis. These equations were all derived using the definition (2.1) of p_{ik} . A fourth formula is obtained by using the alternative definition, as follows. It is shown in [1], Sec. 91 that the analytic function $w_i(z) = u_i + iv_i$ must satisfy the condition

$$w_i'(z) = \frac{P_i(z)}{\left\{ -\prod_{r=1}^{m} (z - a_r)(z - b_r) \right\}^{1/2}}, \tag{2.11}$$

where $P_i(z)$ is a polynomial, of degree not exceeding $m - 2$, with $m - 1$ real coefficients which are uniquely determined by the conditions*:

$$\int_{a_r}^{b_r} w_i'(z)\, dz = \begin{cases} +1, & \text{for } r = j, \\ -1, & \text{for } r = j + 1, \\ 0, & \text{for } r \neq j, \quad r \neq j + 1. \end{cases} \tag{2.12}$$

*It must be remembered that the denominator of $w_i(z)$ changes sign on successive intervals (a_r , b_r).

(It is shown in reference [1] that of the m conditions (2.12) only $m - 1$ are independent.) Now employing the alternative definition of p_{ik} and taking for C_k the doubly-counted kth interval, one easily obtains from (2.11) the result

$$p_{ik} = \mp 2 \int_{b_k}^{a_{k+1}} \frac{P_i(\xi)\, d\xi}{\left\{ \prod_{r=1}^{m} (\xi - a_r)(\xi - b_r) \right\}^{1/2}}, \tag{2.13}$$

where the ambiguity in sign is most easily resolved with the aid of (2.2c) and (2.2d).

Insofar as numerical computations are concerned, it would appear that formulas (2.9) and (2.13) are especially suitable. While (2.13) has an advantage over (2.9) in that integration is necessary only over a single interval, it involves the solution of the system (2.12) for the coefficients of the polynomial $P_i(z)$ and this may become laborious for large values of m. In this case it might prove preferable to employ (2.9), obtaining the function $f_i(x)$ to the desired degree of approximation by the method given in [2], Sec. 6.

3. Extension to more general regions. Since the coefficients of capacitance are conformal invariants, they may be determined by the method described above for any domain which can be mapped conformally onto a slit-domain. A simple example of such a domain is the entire (x, y)-plane slit along a finite number of arcs of a circle (see [1], Sec. 92). By a suitable linear transformation we can map the circumference of this circle onto the x-axis, thus obtaining a slit-domain.

Another domain to which this method applies is a domain bounded externally by one circle and internally by two others. By a suitable linear transformation such a region may always be mapped into a domain with the centers of all three circles collinear (see Fig. 2).

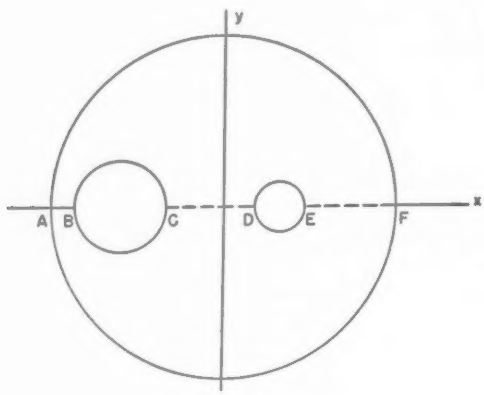

FIG. 2.

Now suppose that the upper half of this domain, which is simply connected, is mapped conformally by an analytic function $\zeta = f(z)$ upon the upper half of the ζ-plane. Then the upper halves of the three circles are mapped into segments of the real axis of the ζ-plane, and the three segments of the axis of symmetry (AB, CD, EF) are mapped into the remainder of the real axis. By the Schwarz reflection principle, the entire domain

of Fig. 2 is then conformally mapped onto the ζ-plane with three slits along the real axis, i.e., upon a slit-domain. A particular case of practical interest in the 'bi-filar shielded conductor'. This case will be discussed in some detail in Sec. 4.

Closely related to the configuration of Fig. 2 is that of the plane with any finite number of circular apertures the centers of which all lie on one line (see Fig. 3). Each

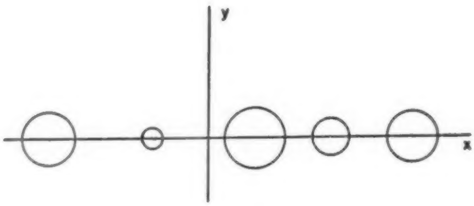

FIG. 3.

half of this region is simply connected. If, as before, we can make a conformal mapping of the upper half of this region upon the upper half of the ζ-plane, then, as in the previous case, the reflection principle immediately gives the mapping of the entire configuration upon the w-plane with collinear slits.

The use of the Schwarz reflection principle to transform a region into one bounded by collinear slits may prove of use in studying the electrostatic field created by an electrified grid.

4. An illustrative example. In the case of the domains described in Sec. 3 it has been shown that the problem of determining the coefficients p_{ik} may be reduced essentially to the problem of finding a mapping function for a simply connected domain, regardless of the connectivity of the original domain. Since the simply connected domain in question is bounded by parts of circles and straight lines, the required mapping can be obtained, in principle, by solving a certain non-linear third-order differential equation which is closely related to a certain linear second order equation (see [3], pp. 198–208). However, in practice this fact is of little use, for these differential equations contain the so-called accessory parameters of the domain, which cannot readily be determined except in the most elementary cases. But for regions of the type under consideration, it might be possible to overcome this difficulty with the help of some of the many approximate conformal mapping techniques that have been developed.

Here, by way of an example, we shall consider a case of the 'bi-filar shielded cable' which can be treated by elementary methods.

Referring to Fig. 4, let the radius of the outer circle be taken equal to unity (this assumption does not reduce the generality), and let the points B, C, D, E correspond to $z = -\beta, -\alpha, \alpha, \beta$ respectively. By the mapping function

$$\zeta = (\alpha\beta)^{1/2}\left(z + \frac{1}{z}\right) \tag{4.1}$$

the interior of the outer circle is mapped upon the exterior of the slit $-2(\alpha\beta)^{1/2} \leq \zeta \leq 2(\alpha\beta)^{1/2}$, and hence the domain bounded by the three circles is mapped onto the ζ-plane minus the aforementioned slit and the images of the two interior circles. In particular, if $\beta \ll 1$, so that the inner circles are small and located close to the center of

the outer circle, then the mapping (4.1) may be approximated in the neighborhood of the inner circles by the linear mapping

$$\zeta = \frac{(\alpha\beta)^{1/2}}{z} \tag{4.2}$$

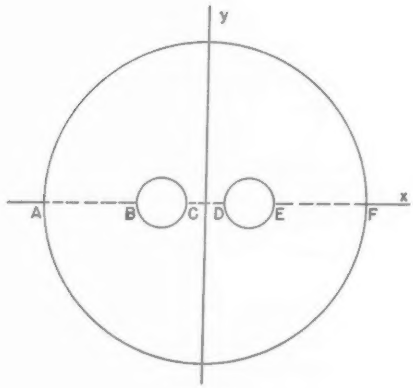

FIG. 4.

which transforms the inner circles into circles which meet the real axis at $\pm(\alpha/\beta)^{1/2}$ and $\pm(\beta/\alpha)^{1/2}$. Hence the domain of Fig. 4 is mapped by (4.1) approximately into the region indicated in Fig. 5. The centers of the circles in Fig. 5 are given by $\zeta = \pm a$,

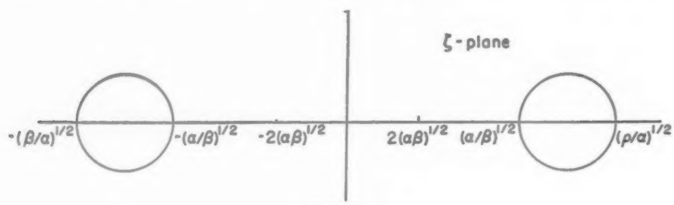

FIG. 5.

where $a = (\alpha + \beta)/[2(\alpha\beta)^{1/2}]$, and the radius of each circle is given by $R = (\beta - \alpha)/[2(\alpha\beta)^{1/2}]$. Since $a^2 - R^2 = 1$, the exterior of these two circles may be mapped upon the exterior of two slits of the real axis by the function (see [3], p. 297, example 9)

$$w = \frac{1 + f[(1+\zeta)/(1-\zeta), \rho]}{1 - f[(1+\zeta)/(1-\zeta), \rho]}, \tag{4.3}$$

where

$$\rho = \frac{(\beta)^{1/2} - (\alpha)^{1/2}}{(\beta)^{1/2} + (\alpha)^{1/2}}$$

and

$$f(t, \rho) = \frac{\rho^2}{t} \frac{\prod_{n=0}^{\infty} (1 + \rho^{8n-2}t^2) \prod_{n=1}^{\infty} (1 + \rho^{8n+2}t^{-2})}{\prod_{n=0}^{\infty} (1 + \rho^{8n+2}t^2) \prod_{n=0}^{\infty} (1 + \rho^{8n+6}t^{-2})} \tag{4.4}$$

(see [3], Chap. VI, Sec. 3, Eqs. (32) and (49)). The circles map onto the twice-counted slits

$$-\frac{1 + L(\rho)}{1 - L(\rho)} \leq w \leq -\frac{1 - L(\rho)}{1 + L(\rho)};$$

and

$$\frac{1 - L(\rho)}{1 + L(\rho)} \leq w \leq \frac{1 + L(\rho)}{1 - L(\rho)}$$

respectively, where

$$L(\rho) = 2\rho \prod_{n=1}^{\infty} \left(\frac{1 + \rho^{8n}}{1 + \rho^{8n-4}} \right)^2. \tag{4.5}$$

The interval $-2(\alpha\beta)^{1/2} \leq \zeta \leq 2(\alpha\beta)^{1/2}$ maps onto the infinite interval

$$| w | \geq \frac{1 - f[\{1 + 2(\alpha\beta)^{1/2}\}/\{1 - 2(\alpha\beta)^{1/2}\}, \rho]}{1 + f[\{1 + 2(\alpha\beta)^{1/2}\}/\}1 - 2(\alpha\beta)^{1/2}\}, \rho]}$$

of the real axis. Thus the domain of Fig. 4 has been approximately mapped into a slit-domain, and the formulas given there may be employed to determine the p_{ik}.

To give a numerical illustration, let $\alpha = 1/20$, $\beta = 1/5$. Then $\rho = 1/3$, and, since the infinite products (4.4) and (4.5) converge rapidly, one obtains very easily

$$\begin{gathered} \frac{1 - L(\rho)}{1 + L(\rho)} = 0.212, \qquad \frac{1 + L(\rho)}{1 - L(\rho)} = 4.726, \\ \frac{1 - f[\{1 + 2(\alpha\beta)^{1/2}\}/\{1 - 2(\alpha\beta)^{1/2}\}, \rho]}{1 + f[\{1 + 2(\alpha\beta)^{1/2}\}/\{1 - 2(\alpha\beta)^{1/2}\}, \rho]} = 8.628 \end{gathered} \tag{4.6}$$

In order to work with more convenient numbers, we employ the transformation

$$Z = \frac{.2}{.212} Z, \tag{4.7}$$

thus obtaining in the Z-plane a slit-domain characterized by the following numbers (see Fig. 1):

$$a_1 = -8.1, \quad b_1 = -4.5, \quad a_2 = -0.2, \quad b_2 = 0.2, \quad a_3 = 4.5, \quad b_3 = 8.1. \tag{4.8}$$

On account of the low connectivity of the domain ($m = 3$) it was decided to employ formula (2.13) to evaluate the p_{ik}. The coefficients p_{22} and p_{21} (the numbering of the boundary components is given in Fig. 4 and Fig. 5, and is in accordance with the numbering given preceding Fig. 1) were determined as follows. The polynomial $P_2(Z)$, which according to the discussion of Sec. 2 is of first degree, was replaced in (2.12) by the *quadratic* polynomial $aZ^2 + bZ + c$ with unknown coefficients a, b, c. The integrals appearing in (2.12) were computed by interpolation (the Tchebyshev five-point formula was used) and the resulting system of linear equations was solved for the coefficients a, b, c, with the following results:

$$a = -9.3 \times 10^{-8}, \qquad b = -1.962, \qquad c = -11.608. \tag{4.9}$$

(The radical appearing in (2.12) was taken positive on the first and third intervals and negative on the second interval.) The extremely small value obtained for a, whose exact value must be zero, serves as an excellent check on the accuracy of the computations. The expression for $P_2(Z)$ thus obtained was then employed in (2.13), with the following results:

$$p_{21} = -2.153, \qquad p_{22} = +3.820. \tag{4.10}$$

By symmetry, $p_{11} = p_{22}$, and now the remaining six coefficients are obtained with the aid of (2.2a) and (2.2b). Thus, the following table of values of the p_{jk} was obtained.

$k \backslash j$	1	2	3
1	+3.820	−2.153	−1.667
2	−2.153	+3.820	−1.667
3	−1.667	−1.667	+3.334

As a check, p_{22} was also computed by means of (2.5) and (2.10), and excellent agreement was obtained.

Acknowledgement. We wish to thank Mr. Charles Kahane, who so capably carried out the computations connected with the illustrative example of Sec. 4.

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GENERAL THEORY OF ELASTIC STABILITY*

BY

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Summary. Some general topics in elastic stability are discussed. In particular, attention is given to the relationship between adjacent-equilibrium-position and energy techniques, to the effects of non-linearity, and to the sensitivity of certain stability problems to the character of the loading.

1. Introduction. In analyzing the stability of an equilibrium state of a particular elastic system, those terms which arise from the equilibrium condition eventually cancel; consequently a number of writers have found it desirable to discuss stability in a general manner, removing these terms once and for all, and directing their attention towards the remaining terms. Usually, a certain equilibrium state is postulated, and one of two criteria is then used to determine the stability of this state. The first criterion states that a structure is unstable if an adjacent equilibrium state exists, whereas the second requires for instability that the over-all potential energy not be a relative minimum.

In setting up these criteria in analytical form, it is recognized that some sort of non-linearity is essential, in order for example to evade the uniqueness theorem of linear elasticity. Such non-linearity may arise either from the geometry of the situation (large displacements or non-linearized boundary conditions) or from the inclusion of higher order terms in the stress-strain law. Although the above uniqueness theorem reason may not be valid (on the grounds that the usual proof of this theorem contemplates the same body configuration for the two supposedly different stress-strain states to be proven identical, and so is not applicable to stability problems in any event), it is nevertheless clear that because of the cancellation of equilibrium terms it is well to include all important higher order terms. The method of incorporating these terms varies widely, as will be seen from a study of treatments by Bryan [1], Southwell [2], Biezeno and Hencky [3], Trefftz [4], Biot [5], Neuber [6], Prager [7], Goodier and Plass [8], and others. Most of the assumptions concerning non-linearity made in these treatments seem rather artificial—for example, Trefftz and Goodier obtain non-linearities by regarding as fundamental a curvilinear coordinate system which moves with the material fibers of the body, and there seems to be little basis for this method. Similarly, Prager uses techniques of superposition which are questionable when dealing with non-linear effects.**

The role of non-linearity has been further complicated as a result of a paper by Goodier [9], in which it is stated that the correct equations for the torsional buckling of a bar can apparently not be obtained by conventional energy techniques. Goodier gives a rather complicated analysis of this problem, using a Trefftz-type method, and his incorporation of non-linearities again appears to be quite arbitrary. In the problem of shell buckling, as discussed for example by Kármán and Tsien [10], it has been suggested

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*Received May 25, 1955; revised manuscript received September 22, 1955.

**Nevertheless, the treatment of Prager is probably the clearest available. He uses the adjacent equilibrium position technique of Biezeno and Hencky, but obtains their results in a much more compact manner. In addition, the effects of inelasticity and thermal gradients are considered, and the final eigenvalue problem is thrown into the form of a variational principle.

that some sort of non-linearity may be partly responsible for the discrepancy between experiment and theory (besides the known effect of initial irregularity).

On the other hand, it is known that the effect of non-linearity in the stress-strain laws governing the usual structural metals is only of the same order of magnitude as the uncertainties in the ordinary elastic constants, and it does not seem physically reasonable that such effects should materially influence practical stability problems. In the present analysis, the stability problem is first analyzed without approximation; an engineering approximation is then obtained by consistently neglecting terms of a certain order of magnitude. Specifically, an arbitrary elastic body in equilibrium under certain loading is considered. An arbitrary virtual displacement is assumed, and the discrepancy between the work done by the loading and the increase in internal energy is calculated by means of the exact stress-strain relationship of non-linear elasticity; the most convenient form of this law is that due to Murnaghan [11]. It is decided that a positive discrepancy is the necessary and sufficient condition for instability, and the analytical consequences of this are worked out. The appropriate engineering approximation is made in the final result, and it is found that the result is different from and simpler than those usually obtained. As a special case, the problem of Goodier [9] is considered and it is found that the correct equations are obtained in a straightforward manner.

Recent discussion by Pflüger [12] and Ziegler [13] have directed attention towards certain fundamental problems in stability. Ziegler has shown by exemplification with non-conservative systems that the result of examining for stability the equations of motion of a perturbed mechanical system do not necessarily coincide with the familiar energy or adjacent-equilibrium-position techniques. Since the equations-of-motion method must be regarded as basic, Ziegler's work gives rise to some doubts as to the usefulness of the other methods. However, for the case of conservative systems (to which we restrict ourselves for the present; plastic buckling will be discussed elsewhere), the equations-of-motion method and the energy method are equivalent (a proof will be found in Whittaker [14]), and so the energy method may be used with confidence. However, the adjacent-equilibrium method and the energy method are certainly not equivalent even for conservative systems. Consider for example an (always elastic) column compressed beyond the buckling load but restrained from buckling; if the constraints are removed the column will buckle despite the absence of an adjacent equilibrium position.

But we can perhaps obtain an equivalence by altering the problem somewhat. Consider an elastic system which is in stable equilibrium under certain loading. As the loading is increased in some manner, the system following an equilibrium path, a point of instability (by the energy criterion) may be reached; does an adjacent equilibrium position exist at this critical point? This is a question of considerable practical significance. For example, in the problem of "tin-canning"—i.e., the problem of stability of a fairly flat shell under lateral pressure, where at a certain critical load the shell tends to snap suddenly through into an entirely different equilibrium position—there seems to be no adjacent equilibrium position (in the conventional Euler-column sense) corresponding to the critical load. Can an adjacent-equilibrium-technique then (as used in practice) give the correct answer to such problems?

It will be shown that the two techniques are indeed identical for the altered problem of the last paragraph (so that, for example, the correct answer in tin-canning problems is that conventionally obtained, and one must look elsewhere for the discrepancy between theory and experiment). In particular, it will be found that the differential equations of

first variation of the general energy principle are precisely the same as the conditions for existence of an adjacent equilibrium position, these conditions being again calculated without approximation.

Another stability topic of considerable interest relates to the precise character of the loading applied to an elastic system. Such effects are not considered in previous general stability analyses, yet have been shown to be important by Tsien [15]. A particularly subtle example is given here in which, for any perturbation, the first-order work done by two alternative types of loading is the same, yet the buckling loads are widely different. Since the problem is a very practical one (buckling of a long cylinder under external pressure), it is clear that the character of the loading must be included in any general stability criterion. The appropriate analysis will be given for the two types of loading of greatest importance, viz., dead loading and pressure loading.

2. Analytical condition or instability with dead loading. Consider an arbitrary elastic body which is initially free from stress (state I). By the application of load or of heat, the body alters position and shape (and achieves state II). The material particle initially at the point (a_1, a_2, a_3) has now moved to (x_1, x_2, x_3), where subscripts are used to distinguish between the usual three fixed Cartesian axes. The ith component of the displacement vector is given by

$$v_i = x_i - a_i \tag{1}$$

and the Lagrangian strain tensor is

$$\eta_{ij} = \tfrac{1}{2}[\partial v_i/\partial a_j + \partial v_j/\partial a_i + (\partial v_s/\partial a_i)(\partial v_s/\partial a_j)], \tag{2}$$

where the summation convention is used (here and in the future) for repeated subscripts.

If U is the internal energy per unit mass, a symmetric function of the nine η_{ij}, of the absolute temperature T (or entropy S), and of position, then by Murnaghan's treatment [11] the Eulerian stress τ_{ij} in state II is given by

$$\tau_{ij} = \rho(\partial U/\partial \eta_{pq})_s(\partial x_i/\partial a_p)(\partial x_j/\partial a_q), \tag{3}$$

where ρ is the density in state II and the partial differentiation of U is to be carried out at constant entropy. It is now required to analyze the stability of the body in its deformed state II. From Sec. 1, the body will be considered stable if for each infinitesimal displacement (compatible with the boundary conditions) the work that would be done by the surface and body forces does not exceed that absorbed as an increase in internal energy.* If this condition is not met, then for some virtual displacement excess energy would be available for use as kinetic energy, and the appropriate displacement will increase in magnitude.

The body force per unit mass, F_i, will be assumed constant (e.g., gravitation). The surface loading, T_i per unit area in state II, is considered to be produced by fixed loads which vary neither in total magnitude nor direction during the trial displacement. Thus, under such "dead" loading, the material particles constituting a portion of the surface in state II will always be subject to the same total surface vector force, irrespective of their orientation or total area, throughout the trial displacement. Consequently, the

*In the equivalent potential energy form, this is the usual energetic stability criterion. We use the above form because of its additional generality; as will be shown elsewhere, it can then be applied to certain non-conservative systems also.

work done by the body and surface forces in a displacement u_i from state II would be, exactly,

$$W = \int \rho F_i u_i \, dV + \int T_i u_i \, dS,$$

where the volume and surface integrals are calculated for state II. Altering to volume integrals and using the equations of equilibrium gives

$$W = \int \tau_{ij}(\partial u_i/\partial x_j) \, dV$$

which, upon substitution from Eq. (3), becomes

$$W = \int (\partial U/\partial \eta_{pq})(\partial x_i/\partial a_p)(\partial u_i/\partial a_q)\rho \, dV. \tag{4}$$

The increase in internal energy is, exactly,

$$\Omega = \int (U' - U)\rho \, dV, \tag{5}$$

where U' denotes the internal energy per unit mass following the displacement u_i, and depends on the temperature of that state as well as on u_i. Note that the volume integral is still calculated for state II (this is allowable because the element of mass, $\rho \, dV$, is invariant).

Consequently, the general condition for stability is that, for each allowable u_i,

$$\int [(\partial U/\partial \eta_{pq})_s(\partial x_i/\partial a_p)(\partial u_i/\partial a_q) - \{U' - U\}]\rho \, dV \leq 0 \tag{6}$$

(in particular the integral vanishes for $u_i = 0$; if it vanishes for some $u_i \neq 0$ but is non-positive for all u_i, the state will be called neutrally stable.)

It is now necessary to calculate U'. Because buckling is usually rapid, it is reasonable to require the displacement u_i to be of an adiabatic character, and we will make this assumption. Had we at this point insisted on an isothermal motion, an entirely analogous calculation (best carried out by use of the Helmholtz free energy function instead of U) could have been made, and the same final results would be obtained in the sequel except that the isothermal rather than the adiabatic elastic constants would appear. Experimentally, the difference between these constants is negligible; then, using the fact that in general the motion u_i of the body would be somewhere between adiabatic and isothermal, it is seen that the particular thermal assumption at this point makes little difference. In any event, we consider for definiteness an adiabatic motion, so that in the power series expansion of U, viz.,

$$U' - U = (\partial U/\partial \eta_{pq}) \, \delta\eta_{pq} + \tfrac{1}{2}(\partial^2 U/\partial \eta_{pq} \, \partial \eta_{ij}) \, \delta\eta_{ij} \, \delta\eta_{pq} + \cdots \tag{7}$$

all partial derivatives are to be calculated for constant entropy and for state II. Using

$$\delta\eta_{ij} = \tfrac{1}{2}[(\partial x_r/\partial a_i)(\partial u_r/\partial a_j) + (\partial x_r/\partial a_j)(\partial u_r/\partial a_i) + (\partial u_r/\partial a_i)(\partial u_r/\partial a_j)] \tag{8}$$

in Eq. (7), and substituting the result into Eq. (6), gives as the condition for stability that

$$\int \rho \, dV[(\partial U/\partial \eta_{ij})(\partial u_r/\partial a_i)(\partial u_r/\partial a_j) + (\partial^2 U/\partial \eta_{ij} \, \partial \eta_{pq}) \, \partial\eta_{ij} \, \delta\eta_{pq} + \cdots]$$

be greater than zero for all non-zero permissible u_i. Alternatively, use of Eq. (3) allows the condition to be written as

$$\int dV[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + \rho(\partial^2 U/\partial\eta_{ij}\,\partial\eta_{pq})\,\delta\eta_{ij}\,\delta\eta_{pq} + \cdots] > 0. \tag{9}$$

Except for pathological cases, only second-order terms need be considered, and the criterion for stability becomes

$$\int dV[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + \rho(\partial^2 U/\partial\eta_{ij}\,\partial\eta_{pq})(\partial x_r/\partial a_i)(\partial x_s/\partial a_p)(\partial u_r/\partial a_j)(\partial u_s/\partial a_q)] > 0. \tag{10}$$

Here, all quantities are calculated for state II. For adiabatic virtual displacements, this criterion is exact, and must be used wherever non-linearity of the stress-strain law is essential.

3. Engineering approximation. Isotropic media. The second term in Eq. (10) may be calculated by means of a power series expansion in η_{ij} in terms of the various derivatives of U evaluated for state I. Using the subscript "0" to indicate state I,

$$\partial^2 U/\partial\eta_{ij}\,\partial\eta_{pq} = (\partial^2 U/\partial\eta_{ij}\,\partial\eta_{pq})_0 + (\partial^3 U/\partial\eta_{ij}\,\partial\eta_{pq}\,\partial\eta_{rs})_0\eta_{rs} + \cdots . \tag{11}$$

For structural metals, the magnitude of the second term in Eq. (11) is generally smaller than the uncertainty in the experimental value of the first term (the first term represents the usual elastic constants, and the second and following terms represent non-linear elastic effects). Consequently, it is reasonable to approximate the second term coefficient by

$$\rho_0\,(\partial^2 U/\partial\eta_{ij}\,\partial\eta_{pq})_0\ ,$$

where the density in state II has also been replaced by the density in state I. This term is recognized as the conventional (adiabatic) elastic coefficient and will be denoted by c^0_{ijpq}. Then the second term may be written

$$[c^0_{ijpq}(\partial x_r/\partial a_i)(\partial x_t/\partial a_j)(\partial x_s/\partial a_p)(\partial x_m/\partial a_q)](\partial u_r/\partial x_t)(\partial u_s/\partial x_m). \tag{12}$$

Now the deformation (although not the displacement) between states I and II is assumed small; this means that the partial derivatives inside the square bracket of (12) represent, within the approximation being made, a pure rotation. But the quantities c^0_{ijpq} form a Cartesian tensor, so that the quantity in square brackets reduces simply to the elastic coefficients for the orientation of state II, i.e., to c_{rtsm}. Thus the stability condition, Eq. (10), becomes

$$\int dV[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + c_{rtsm}e_{rt}e_{sm}] > 0, \tag{13}$$

where $e_{ij} = \frac{1}{2}[(\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)]$ and where the symmetry property of c_{rtsm} has been used. For isotropic media,

$$c_{rtsm} = G\left[\delta_{rs}\delta_{tm} + \delta_{ts}\delta_{rm} + \frac{2\sigma}{1-2\sigma}\,\delta_{rt}\delta_{sm}\right], \tag{14}$$

where G is the shear modulus and σ is Poisson's ratio. Use of this relation gives the stability criterion as

$$\int dV\left[\tau_{pq}(\partial u_r/\partial x_p)(\partial u_r/\partial x_q) + 2G\left\{e_{sm}e_{sm} + \frac{\sigma}{1-2\sigma}\,e_{tt}e_{mm}\right\}\right] > 0. \tag{15}$$

The only terms which have been neglected in the derivation of Eq. (15) are those which are of higher order in powers of η_{ij} .

4. Euler column. Before proceeding with the general theory, it is worthwhile to consider a simple example. Let a slender column of length L and cross-sectional area A be placed so that its neutral axis coincides with the x_3 axis, and so that it may buckle in the $x_1 - x_3$ plane only, the ends being restrained from lateral motion. A total load P is applied to the end of the column, producing a stress of $\tau_{33} = -(P/A)$, all other $\tau_{ij} = 0$. We now use Eq. (15), and see how large P must be, for certain trial displacements, before the left-hand side of Eq. (15) becomes negative; such a situation would correspond to buckling. Since the trial displacement will usually not be the exactly best ones for this purpose, the buckling load obtained in this manner will always be too high; this remark clearly holds in general also and is not restricted to the Euler column case (see Ref. [8]). In fact, the buckling load P_c is given by

$$(P_c/2GA) = \min_{\{u_i\}} \frac{\int \{e_{sm}e_{sm} + (\sigma/1 - 2\sigma)e_{tt}e_{mm}\}\, dV}{\int (\partial u_r/\partial x_3)(\partial u_r/\partial x_3)\, dV}. \tag{16}$$

Whenever P exceeds P_c , Eq. (15) shows that the column is unstable.

Let us choose a trial displacement, being guided in our choice by the tendency of plane cross sections to remain plane and perpendicular to the neutral axis during bending:

$$\begin{aligned} u_1 &= u(x_3), \\ u_2 &= 0, \\ u_3 &= -x_1 u'(x_3), \end{aligned} \tag{17}$$

where $u(x_3)$ is an arbitrary function of x_3 which vanishes at $x_3 = 0$, L, and where $u'(x_3)$ is its derivative. A straightforward calculation using Eq. (15) gives that

$$-(P/A)\left[A \int_0^L (u')^2 + I \int_0^L (u'')^2\right] + 2G\left(\frac{1-\sigma}{1-2\sigma}\right)I \int_0^L (u'')^2 < 0 \tag{18}$$

for instability, where I is the appropriate moment of inertia of the cross section. Because $(P/A) \ll G$, the second term in the square bracket is omitted and we obtain

$$\begin{aligned} P_c &= 2GI\left(\frac{1-\sigma}{1-2\sigma}\right) \min \frac{\int_0^L (u'')^2}{\int_0^L (u')^2} \\ &= 2GI\left(\frac{1-\sigma}{1-2\sigma}\right)(\pi/L)^2. \end{aligned} \tag{19}$$

This answer is too high, because

$$2G\left(\frac{1-\sigma}{1-2\sigma}\right) > E$$

so that the displacement (17) is deficient in some respect. The deficiency lies in the fact that u_1 and u_2 do not contain terms allowing for lateral expansion of the column during bending. Actually, instead of setting $e_{11} = e_{22} = 0$ in Eq. (15), we should more correctly have set $e_{11} = e_{22} = -\sigma e_{33}$. A simple calculation shows that the incorporation of such terms does not materially alter the first term in Eq. (15). A suitable altered displace-

ment would in fact be

$$\begin{aligned} u_1 &= u + \frac{\sigma}{2}(x_1^2 - x_2^2)u'', \\ u_2 &= \sigma x_1 x_2 u'', \\ u_3 &= -x_1 u'. \end{aligned} \tag{20}$$

If this displacement is inserted into Eq. (15), and the magnitude of various terms examined (which is most easily done by assuming that u is not far removed from that u used to minimize Eq. (19), viz., $\sin(\pi x/L)$), it is found that the first term of Eq. (15) is essentially unaltered, whereas the second becomes (very closely) Ee_{33}^2. Then a similar calculation to that of Eq. (19) gives

$$P_e = EI(\pi/L)^2.$$

It will be noted in Eq. (15) that the second term is the familiar strain energy term, so that the first term must in a sense represent work done by the loading. It is therefore not surprising that minor transverse alterations in the u_i (these alterations incidentally vanishing on the neutral axis) do not affect to any extent the value of the first term. This is a rather useful point to note, because it means that simple displacements of the type (17) may be used in many column problems, provided only that e_{11} and e_{22} are set equal to $(-\sigma e_{33})$ in Eq. (15).

5. Flexural buckling. Goodier [9] has examined the use of energy techniques in the flexural buckling of a twisted bar. Since his analysis is geometrically complicated and physically questionable it is worthwhile to show that the correct equations are obtained by use of Eq. (15) in a routine manner. Since the only question is as to whether or not certain terms occur, it is only necessary to consider a simple special case—that of a circular cylinder. If the central line coincides with the x_3-axis and if the angle of twist per unit length is θ, the stresses are

$$\tau_{13} = -G\theta x_2, \qquad \tau_{23} = G\theta x_1.$$

Assume a displacement of the form

$$\begin{aligned} u_1 &= u - \beta x_2, \\ u_2 &= v + \beta x_1, \\ u_3 &= -x_1 u' - x_2 v', \end{aligned}$$

where u, v, β are functions of x_3. (If the cylinder were non-circular, a term involving the warping function multiplied by β' should be adjoined to u_i.) Then using the technique of Sec. 4, condition (15) requires for stability

$$2G\theta \int_0^L (-u'v''I_1 + v'u''I_2) + GI_0 \int_0^L (\beta')^2 + E \int_0^L [I_2(u'')^2 + I_1(v'')^2] > 0$$

and minimizing* this expression yields

$$EI_2 u''' + Mv'' = 0, \qquad EI_1 v''' - Mu'' = 0, \qquad \beta'' = 0.$$

*That minimization is the appropriate procedure will be shown subsequently.

With the appropriate boundary conditions, these coincide with the final results of Goodier. (Note that I_1 and I_2 are defined in an opposite way to that of Goodier. Here I_1 is defined as being about the x_1-axis, i.e., $\int x_2^2\, dA$.)

6. Curvilinear coordinates. Buckling of a cylinder under dead load. Very often, the appropriate coordinate system is not Cartesian; in such cases it is useful to have available a more general formulation of Eq. (15). Let the differential element of distance be given by

$$ds^2 = h_1^2\, dy_1^2 + h_2^2\, dy_2^2 + h_3^2\, dy_3^2\ ,$$

where h_1, h_2, h_3 are functions of the three curvilinear coordinates y_1, y_2, y_3. Then denoting by τ_{ij} the curvilinear stress components and by u_i the curvilinear virtual displacement component (i.e., in the parametric direction of y_i), the first term of Eq. (15) may be shown by direct calculation to become

$$\sum_{p,q,r,m} \left[\frac{\tau_{pq}}{h_p h_q}\left\{u_{r,p}u_{r,q} + \frac{u_p u_q}{h_r^2}\, h_{p,r}h_{q,r} + \left(\frac{u_r u_m}{h_r h_m}\, \delta_{pq} h_{p,m} h_{p,r} + \frac{2u_m}{h_m}\, h_{q,m}u_{q,p} - \frac{2u_q}{h_r}\, u_{r,p}h_{q,r} - \frac{2u_m u_p}{h_m h_q}\, h_{p,q}h_{q,m}\right)\right\}\right], \tag{21}$$

where a comma indicates differentiation with respect to the appropriate y_i—thus $u_{r,p}$ means $(\partial u_r/\partial y_p)$.

The form of the second term is unaltered, but e_{ij} must now be interpreted as a curvilinear strain component, perhaps most conveniently given by

$$e_{ij} = \frac{1}{2} \sum_s \left[\frac{h_j}{h_i}\frac{\partial}{\partial y_i}\left(\frac{u_j}{h_j}\right) + \frac{h_i}{h_j}\frac{\partial}{\partial y_j}\left(\frac{u_i}{h_i}\right) + 2\delta_{ij}\,\frac{u_s}{h_s h_i}\,\frac{\partial h_i}{\partial y_s}\right]. \tag{22}$$

Consider for example a long thin circular shell of mean radius R, under the action of an external pressure P of the present dead-loading (see Sec. 3) type. As cylindrical coordinates, choose $y_1 = x$ along the axis of the shell, $y_2 = \theta$, the polar angle, and $y_3 = r$, the radial distance from the central axis. Then

$$ds^2 = dy_1^2 + y_3^2\, dy_2^2 + dy_3^2\ .$$

Choose

$$u_1 = 0,$$

$$u_2 = v + \frac{z}{R}\,(v - w'),$$

$$u_3 = w,$$

where v, w are functions of θ representing the motion of the central surface of the shell, and $z = r - R$. From shell theory, it is known that displacements of this type are suitable for calculating all strains except e_{13}, e_{23}, e_{33}. The former two are conventionally negligible, and the latter is usually calculated by assuming the induced τ_{33} stresses to be much smaller than the bending stresses in the shell. Then the e_{ij} to be used in Eq. (15) are

$$e_{11} = e_{12} = e_{13} = e_{23} = 0,$$

$$e_{22} = \frac{v'}{R} + \frac{w}{R} - \frac{z}{R^2}\,(w + w''),$$

$$e_{33} = -\frac{\sigma}{1-\sigma}\, e_{22}\ ,$$

where a minor approximation has been made. Using Eq. (21), the stability condition (15) becomes that

$$\int dV\left\{\left(-\frac{PR}{tr^2}\right)[(u_{2,2}+u_3)^2+(u_2-u_{3,2})^2]+\frac{E}{1-\sigma^2}\,[e_{22}^2]\right\}>0,$$

where t is the thickness of the shell. Because $(PR)/t \ll E$, the first term may be submerged in the last to give, approximately,

$$\int dV\left\{-\frac{P}{tR}\,[(v-w')+\frac{z}{R}(v-w')]^2+\frac{E}{1-\sigma^2}\left[\frac{v'}{R}+\frac{w}{R}-\frac{z}{R^2}(w+w'')\right]^2\right\}>0. \quad (23)$$

Taking a unit length of cylinder and integrating over the cross-sectional area gives

$$P_c=\frac{E}{(1-\sigma^2)R}\min\frac{t\int_0^{2\pi}(v'+w)^2+(t^3/12R^2)\int_0^{2\pi}(w+w'')^2}{\int_0^{2\pi}(v-w')^2}$$

which is obtained (very closely) by setting $v' = -w$ and $v = \sin 2\theta(v = \sin\theta$ would correspond to rigid body motion). Then

$$P_c=\frac{4EI}{(1-\sigma^2)R^3}, \quad (24)$$

where $I = t^3/12$. Since the usual result is (3/4) of this, it is clear that the assumption of dead loading has materially altered the critical load. Load-type sensitivity has been remarked for this problem by Stevens [16] and more generally by Tsien [15]; we consider it here to exemplify the manner in which the stability criterion will be generalized.

7. Pressure loading. We return now to the general theory of Sec. 2, and examine the effects of different types of loading. Firstly, it is clear that forces exerted by fixed constraints (pin joints, etc.) are in general included in the theory of Sec. 2, for even though such forces may alter in direction and magnitude during a trial displacement, the appropriate component of u_i at this point of application is zero. If secondly, however, some of the surface tractions are not of the dead-loading type, then additional terms must in general be adjoined to Eq. (15). We consider here only the practically most important such forces, viz., pressure-type forces, for which the force applied to a given portion of the surface of the body varies in such a manner as to remain always perpendicular to that portion and so as always to maintain the same magnitude per unit area. Further, the system is still assumed conservative, so that the total work done by these pressure forces is independent of the path. If then a pressure P acts on a portion S_P of the surface, the work done in the trial displacement u_i can be calculated by allowing the intermediate displacement to grow at a constant rate—i.e., if t is time, let the displacement at time t be $(u_i t)$ and calculate the work done from $t = 0$ to $t = 1$. This work, w_1, is

$$W_1=\int_0^1 dt\int (dS_P)_t\left[\frac{d}{dt}(u_i t)\cdot(-P)\cdot(n_i)_t\right],$$

where the subscript t denotes evaluation at time t. But

$$(n_i)_t(dS_P)_t=\frac{1}{2}e_{ijk}e_{rpq}\frac{\partial(x_j+u_jt)}{\partial x_p}\frac{\partial(x_k+u_kt)}{\partial x_q}n_r\,dS_P$$

so that

$$W_1 = \int_0^1 dt \int dS_P\left[\frac{1}{2} e_{ijk}e_{rpq}\{\delta_{ip} + u_{i,p}t\}\{\delta_{kq} + u_{k,q}t\}n_r\right](-Pu_i),$$
$$= -P \int dS_P\left[n_i u_i + \frac{1}{2}(n_i u_{k,k} u_i - n_k u_{k,i} u_i) + \frac{1}{6} e_{ijk}e_{rpq}u_{j,p}u_{k,q}u_i\right]. \tag{25}$$

In this exact expression, the first term would already have been included if P had been treated as a dead load; consequently the additional work done is that due to the remaining terms. Again we omit terms of third order in u_i [see Eq. (10)], and remembering that a factor of -2 was incorporated into the derivation of Eq. (10), the term that must be adjoined to Eq. (10) is

$$\int dS_P P[n_i u_{k,k} u_i - n_k u_{k,i} u_i]. \tag{26}$$

Considering again the problem of Sec. 6, the term to be added to Eq. (23) is easily seen to be

$$P \int_0^{2\pi} (w^2 + wv' - vw' + v^2)\, d\theta$$

and instead of Eq. (24) we obtain

$$P_c = \frac{3EI}{(1 - \sigma^2)R^3} \tag{27}$$

which is the conventional result.

8. Adjacent-equilibrium-position method. It is now proposed to set up in analytical form the condition that an adjacent equilibrium position should exist. Using the notation of Sec. 2, and denoting quantities in the perturbed state by primes, the stresses following the virtual displacement u_i will be

$$\tau'_{ij} = \rho'\left(\frac{\partial U}{\partial \eta_{pq}}\right)' \frac{\partial x'_i}{\partial a_p} \frac{\partial x'_j}{\partial a_q}, \tag{28}$$

where $x'_i = x_i + u_i$. The partial derivatives of U will as before be calculated at constant entropy. Expanding the energy term in a power series gives

$$\left(\frac{\partial U}{\partial \eta_{pq}}\right)' = \frac{\partial U}{\partial \eta_{pq}} + \frac{\partial^2 U}{\partial \eta_{pq}\, \partial \eta_{rs}} \delta\eta_{rs} + \frac{1}{2} \frac{\partial^3 U}{\partial \eta_{pq}\, \partial \eta_{rs}\, \partial \eta_{lm}} \delta\eta_{rs}\, \delta\eta_{lm} + \cdots, \tag{29}$$

where the partial derivatives on the right-hand side of Eq. (29) are evaluated for state II, and where $\delta\eta_{ij}$ is given by Eq. (8). Inserting this result into Eq. (28), replacing x'_i by $x_i + u_i$, using Eq. (3), and neglecting all higher order terms (a process which by virtue of Secs. 2 and 3 is considered legitimate) gives eventually

$$\tau'_{ij} = \frac{\rho'}{\rho} \tau_{ij} + \tau_{is} u_{j,s} + \tau_{rj} u_{i,r} + \rho \frac{\partial^2 U}{\partial \eta_{pq}\, \partial \eta_{rs}} \left(\frac{\partial x_i}{\partial a_p} \frac{\partial x_j}{\partial a_q} \frac{\partial x_t}{\partial a_s} \frac{\partial x_m}{\partial a_r} u_{t,m}\right), \tag{30}$$

where a comma indicates partial differentiation with respect to x_i. Then using

$$\frac{\partial}{\partial x'_i} = [\delta_{si} - u_{s,i}] \frac{\partial}{\partial x_s}$$

(within higher order terms) the equation

$$\frac{\partial}{\partial x_i'}(\tau_{ij}') + \rho' F_i = 0$$

becomes

$$\tau_{rj,j} u_{i,r} + \tau_{rj} u_{i,rj} + \left[\rho \frac{\partial^2 U}{\partial \eta_{pq}\, \partial \eta_{rs}} \frac{\partial x_i}{\partial a_p} \frac{\partial x_j}{\partial a_q} \frac{\partial x_t}{\partial a_s} \frac{\partial x_m}{\partial a_r} u_{t,m}\right]_{,j} = 0. \tag{31}$$

In addition to Eqs. (31), a boundary condition must be satisfied. For the case of dead loading, the condition is that

$$T_i'\, dS' = T_i\, dS$$

and using the fact that

$$n_i'\, dS' = (\rho/\rho') \frac{\partial x_r}{\partial x_i'} n_r\, dS$$

gives as this condition

$$\left(\tau_{pq} u_{r,p} + \rho \frac{\partial^2 U}{\partial \eta_{ij}\, \partial \eta_{pf}} \frac{\partial x_r}{\partial a_p} \frac{\partial x_s}{\partial a_i} \frac{\partial x_q}{\partial a_f} \frac{\partial x_l}{\partial a_j} u_{s,l}\right) n_q = 0. \tag{32}$$

Similarly, the boundary condition for that part of the surface where pressure forces act is

$$T_i'\, dS' = -Pn_i'\, dS',$$

whence it is found that Eq. (32) should be altered for this portion of the surface by adding to the left-hand side the term

$$P(\delta_{qr} u_{s,s} - u_{q,r}) n_q\, . \tag{33}$$

9. Relation between the two methods. It has been remarked in Sec. 1 that the methods of Sec. 2 and Sec. 8 can at best be equivalent only for special situations, such as at points where an originally stable structure first becomes unstable. Consider therefore a structure which follows some stable equilibrium path as the load alters. The path will remain stable as long as the second order variation in potential energy is positive definite (vanishing only for zero displacement). Consequently, trouble can occur only at points where this second order variation [essentially the left-hand side of Eq. (10)] vanishes for non-zero displacements. Such a situation of neutral stability will in practice be followed by unstable equilibrium states as the load is further increased (see Poincaré [17]); we therefore investigate the condition under which the left-hand side of Eq. (10) first vanishes for non-zero displacements. Since it always vanishes for zero displacements, this condition is equivalent to requiring the minimum of Eq. (10) to be attained at non-zero u_i as well as at zero u_i, and this eigenvalue problem is that obtained by setting the first variation of Eq. (10) equal to zero.

The result of doing this is easily seen to be the same as Eq. (31), with the natural boundary condition (32). If pressure forces act on a portion of the surface, the result of a variation of Eq. (26) must be adjoined to the natural boundary conditions. This is not quite straightforward, because the condition that the pressure loading be conservative has not been explicitly stated (without such accessory conditions, the exact pressure

work would depend on the path). The appropriate condition turns out to be that on the boundary of that portion of the surface where pressure forces act,

$$\int e_{rpq} u_p \, \delta u_q \, dx_r = 0 \tag{34}$$

for all variations δu_q . This condition would for example certainly be satisfied if the pressure surface completely enclosed the body, for then the path length would vanish. Alternatively the restraints on u_i may ensure satisfaction of Eq. (34)—as in the case of a hemispherical shell set on a rigid frictionless plane surface and subjected to external pressure.

Using Stokes' theorem on Eq. (34) gives

$$\int dS_P[(u_k \, \delta u_s)_{,s} - (u_s \, \delta u_k)_{,s}]n_k = 0$$

and if this is used in calculating the variation of Eq. (26), Eq. (33) is indeed obtained. It thus follows that at points where Eq. (10) first attains the minimum value of zero for non-zero u_i , an adjacent equilibrium position exists, and in this sense the two methods are equivalent.

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DISPERSION OF MASS BY MOLECULAR AND TURBULENT DIFFUSION: ONE-DIMENSIONAL CASE*

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1. Introduction. If a solute is placed in a solvent in turbulent motion, it is dispersed both by molecular and turbulent diffusion. We derive here, in a one-dimensional case, a formula for the mean concentration of solute, as a function of x and t, in terms of its initial distribution, the coefficient of molecular diffusion, and the statistical characteristics of the turbulent velocity field.

The one-dimensional problem for the infinite domain is treated as follows. Using the initial condition expressed by Eq. (2) below, an initial value problem (for the concentration of solute) is formulated for the diffusion-convection differential equation, Eq. (1), which contains a rather arbitrary convection velocity $v(x, t)$ which is a function of x and t. The solution is obtained in the form of a perturbation series, by perturbing with respect to the "magnitude" of the convection velocity, and the nth order term of this series involves, besides the initial data and the coefficient of diffusion, an n-fold product of convection velocity factors, each evaluated at a different point and time. This solution is found to be valid at small dispersion times or for small intensities of turbulence.

Now let the convection velocity be a *random* (though still continuous and sufficiently differentiable) function of x and t; that is, let it be a member of an ensemble of functions, where the ensemble represents the turbulent velocity field. (This is the approach taken in the mathematical theory of turbulence.) When an ensemble average is taken of each term of the previously obtained perturbation series, there results a power series representation for the *mean* concentration in which the nth order term contains the nth order correlation coefficient of the turbulent velocity field. This technique, of introducing random functions into the theory of partial differential equations, is not a new one. Kampé de Fériet, for instance, has treated several classical initial value problems with random initial data (see [1, 2, 3][1]).

The initial value problem is solved in Sec. 2. In Sec. 3 a physical interpretation is given of a sufficient condition for uniform convergence of the perturbation series solution. In Sec. 4 turbulence is introduced, in the manner indicated above, and in Sec. 5 the following example is considered: dispersion at small times for initial distributions and space correlation coefficients of Gaussian type.

The investigation described in this paper is now being extended in two directions. On the one hand, the treatment presented here will be applied to the corresponding problems in two and three dimensions. On the other hand, the formal operations employed here, especially those involved in the probability approach to turbulence, must be justified.

The author is extremely grateful to Professor J. Kampé de Fériet for suggesting this

*Received May 31, 1955. This work was supported by the Bureau of Ordnance, Department of the Navy, under Contract NOrd 7386, and was presented orally to the American Physical Society April 29, 1955, in Washington, D. C.

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[1]Numbers in brackets refer to the bibliography listed at the end of the paper.

line of inquiry. Also, he wishes to thank his colleagues, O. J. Deters and R. J. Rubin, for several helpful suggestions.

2. Solution by perturbation of the initial value problem for the diffusion-convection differential equation. The one-dimensional equation governing molecular diffusion in the presence of a convection velocity is

$$\frac{\partial s}{\partial t} - D\frac{\partial^2 s}{\partial x^2} = -\frac{\partial(sv)}{\partial x} \tag{1}$$

(see [4], vol. II, p. 593), where $s(x, t)$ = concentration (mass per unit length) of dispersing matter, D = (constant) coefficient of molecular diffusion, $v(x, t)$ = convective velocity, assumed to depend on x and t in such a way that v, $\partial v/\partial x$ and $\partial^2 v/\partial x^2$ are continuous in x and t together.

For later purposes we assume further that $v(x, t)$, $v_x(x, t)$ and $v_{xx}(x, t)$ are uniformly bounded in an infinite strip $-\infty < x < \infty$, $0 \leq t \leq t_0$, i.e., that there are finite positive numbers v^m, v_x^m, and v_{xx}^m, which are the least upper bounds of $|v(x, t)|$, $|v_x(x, t)|$ and $|v_{xx}(x, t)|$, respectively, for $-\infty < x < \infty$, $0 \leq t \leq t_0$.

Consider the initial value problem consisting of Eq. (1) and the initial condition

$$s(x, 0) = f(x) \qquad (-\infty < x < \infty) \tag{2}$$

in the domain $-\infty < x < \infty$, $0 \leq t \leq t_0$. The function $f(x)$ is a given non-negative-valued function of x, twice continuously differentiable and uniformly bounded for $-\infty < \xi < \infty$. We introduce the dimensionless quantities

$$\begin{gathered} \tau = t/T, \qquad \tau_0 = t_0/T, \qquad \xi = x/(TD)^{1/2}, \qquad \sigma(\xi, \tau) = (TD)^{1/2}s(x, t)/Q, \\ \phi(\xi) = (TD)^{1/2}f(x)/Q, \qquad \Omega = (T/D)^{1/2}V, \qquad \omega(\xi, \tau) = v(x, t)/V, \end{gathered} \tag{3}$$

where T is some characteristic time for the diffusion process, $Q = \int_{-\infty}^{\infty} f(x)\,dx$ is the total mass present in the system at $t = 0$, and V is some measure of the magnitude of the convective velocity. In the case of a specific (i.e., non-random) convection velocity $v(x, t)$ we could let $V = v^m$, while in the turbulent case we shall let V equal the root mean square turbulent velocity.

In terms of these dimensionless quantities, the initial value problem becomes

$$\frac{\partial \sigma}{\partial \tau} - \frac{\partial^2 \sigma}{\partial \xi^2} = -\Omega\frac{\partial(\sigma\omega)}{\partial \xi} \qquad (-\infty < \xi < \infty, 0 \leq \tau \leq \tau_0) \tag{4}$$

$$\sigma(\xi, 0) = \phi(\xi) \qquad (-\infty < \xi < \infty). \tag{5}$$

It follows from Eq. (3) and the definition of Q that

$$\int_{-\infty}^{\infty} \phi(\xi)\,d\xi = 1. \tag{6}$$

Assuming that the solution of (4) is expressible as a power series in Ω:

$$\sigma(\xi, \tau) = \sum_{i=0}^{\infty} \Omega^i \sigma_i(\xi, \tau), \tag{7}$$

we find, after substituting this series for $\sigma(\xi, \tau)$ in (4), collecting all terms on the left hand side of the equation, and setting the coefficient of each power of Ω equal to zero, that

$$\frac{\partial \sigma_0}{\partial \tau} - \frac{\partial^2 \sigma_0}{\partial \xi^2} = 0, \tag{8}$$

$$\frac{\partial \sigma_i}{\partial \tau} - \frac{\partial^2 \sigma_i}{\partial \xi^2} = -\frac{\partial}{\partial \xi}(\omega \sigma_{i-1}) \qquad i = 1, 2, \cdots \tag{9}$$

The solution to (8) satisfying the initial condition (5) is

$$\sigma_0(\xi, \tau) = \int_{-\infty}^{\infty} \phi(\xi')\Gamma(\xi - \xi', \tau)\, d\xi', \tag{10}$$

where

$$\Gamma(\xi, \tau) = (4\pi\tau)^{-1/2} \exp(-\xi^2/4\tau) \tag{11}$$

is the so-called fundamental solution of the heat-conduction equation (8). Furthermore, $\sigma_0(\xi, \tau)$ satisfies the normalization condition

$$\int_{-\infty}^{\infty} \sigma_0(\xi, \tau)\, d\xi = 1. \tag{12}$$

It can be shown, by the use of Duhamel's theorem (see [6]) and an integration by parts, that when $\omega(\xi, \tau)$ is uniformly bounded in $-\infty < \xi < \infty$, $0 \leq \tau \leq \tau_0$, a solution of (9) which vanishes at $\tau = 0$ is given by

$$\sigma_i(\xi, \tau) = -\int_0^{\tau} \frac{d\tau'}{[\pi(\tau - \tau')]^{1/2}} \int_{-\infty}^{\infty} \omega\sigma_{i-1} e^{-\alpha^2}\alpha\, d\alpha, \tag{13}$$

where ω and σ_{i-1} are evaluated at $[\xi + 2(\tau - \tau')^{1/2}\alpha, \tau']$. Thus when σ_0 and $\sigma_{i-1}(i = 1, 2, \cdots)$ are given by (10) and (13) respectively, the infinite series (7) solves the initial value problem consisting of Eqs. (4) and (5) (provided this series and those formed by differentiating it term by term converge uniformly in the infinite strip under consideration; this question will be discussed in the next section).

If in (13) we introduce the new variable of integration $\xi' = \xi + 2(\tau - \tau')^{1/2}\alpha$, we get

$$\sigma_i(\xi, \tau) = -\int_0^{\tau} d\tau' \int_{-\infty}^{\infty} \omega(\xi', \tau')\sigma_{i-1}(\xi', \tau') \frac{\partial}{d\xi} \Gamma(\xi - \xi', \tau - \tau')\, d\xi', \tag{14}$$

where $\Gamma(\xi, \tau)$ is given in (11); ($\partial/\partial\xi$ means differentiation with respect to the first argument). Then since Γ vanishes at $\xi = \pm\infty$, it can be shown, by inverting the order of integration, that $\int_{-\infty}^{\infty} \sigma_i(\xi, \tau)d\xi = 0$ for $i = 1, 2, 3, \cdots$. This result, together with Eq. (12), implies that solution (7) conserves mass.

3. Sufficient condition for uniform convergence and its physical interpretation. It can be shown that series (7) and the series obtained by differentiating (7) term by term converge uniformly in the strip $-\infty < \xi < \infty$, $0 \leq \tau \leq \tau_0$ when

$$4\left(\frac{\tau_0}{\pi}\right)^{1/2} \Omega\left\{ \underset{\substack{-\infty<\xi.<\infty \\ 0\leq\tau'\leq\tau_0}}{\text{l.u.b.}} [\,|\,\omega(\xi', \tau')\,| + |\,\omega_\xi(\xi', \tau')\,| + |\,\omega_{\xi\xi}(\xi', \tau')\,|\,]\right\} < 1. \tag{15}$$

Using Eqs. (3) to translate this condition into physical variables, we get $4(t_0/D\pi)^{1/2}(v^m + v_x^m + v_{xx}^m) < 1$, where v^m, v_x^m, and v_{xx}^m are defined following Eq. (1).

This means that the power series is a valid representation of the solution of the initial value problem in the time interval $0 \leq t \leq t_0$ either (1) when for t_0 given arbitrarily, the bound on the convection velocity and its first two x-derivatives is sufficiently small, or (2) when for an arbitrary uniformly bounded convection field (twice continuously differentiable with respect to x), t_0 is sufficiently small. In connection with case 2 (antici-

pating the next section by taking a turbulent velocity field as our convection field) it has been noted in [5], p. 99, in considering dispersion from a point source, that "the effects of molecular agitation on the dispersion are not always negligible as compared with the effects of turbulence; indeed, when the dispersion process starts, the former effect is greater than the latter."

4. Application to turbulence. Now let $\omega(\xi, \tau)$ be a *random* function (though still possessing all the regularity properties with respect to ξ and τ previously assumed for our convection velocity) with $\langle\omega(\xi, \tau)\rangle = 0$ at every point (ξ, τ). The case of a uniform non-zero mean velocity, $\langle\omega\rangle$, can be reduced to the present case by introducing a new space variable, $\eta = \xi - \langle\omega\rangle\tau$. The angle brackets denote an ensemble or stochastic average. Thus we regard the convection (i.e., large-scale motion as compared with the molecular agitation) as arising from a turbulent velocity field with zero mean velocity. It is further assumed that $\langle\omega^2(\xi, \tau)\rangle$ is a constant independent of ξ and τ. For convenience we set $\langle\omega^2\rangle \equiv 1$, which can be done by letting $V = \langle v^2\rangle^{1/2}$, so that Ω is now proportional to the root mean square turbulent velocity. (See the discussion following Eq. (3).)

Rather than σ itself, we are now interested in the *mean* concentration, $\langle\sigma\rangle$, which we propose to find by averaging each of the terms in the series (7). Since σ_0, given by (10), does not contain ω, then $\langle\sigma_0\rangle = \sigma_0$. From (14)

$$\sigma_1(\xi, \tau) = -\int_0^\tau d\tau' \int_{-\infty}^{\infty} \omega(\xi', \tau')\sigma_0(\xi', \tau') \frac{\partial}{\partial\xi} \Gamma(\xi - \xi', \tau - \tau')\, d\xi',$$

so that σ_1 is linear in ω, and since $\langle\omega\rangle \equiv 0$, it follows that

$$\langle\sigma_1(\xi, \tau)\rangle \equiv 0. \tag{16}$$

Again, if we use (14) and the preceding expression for σ_1, we have

$$\sigma_2(\xi, \tau) = \int_0^\tau d\tau' \int_{-\infty}^{\infty} \omega(\xi', \tau') \frac{\partial}{\partial\xi} \Gamma(\xi - \xi', \tau - \tau')\, d\xi'$$
$$\times \int_0^{\tau'} d\tau'' \int_{-\infty}^{\infty} \omega(\xi'', \tau'')\sigma_0(\xi'', \tau'') \frac{\partial}{\partial\xi} \Gamma(\xi' - \xi'', \tau' - \tau'')\, d\xi''.$$

Thus, σ_2 involves a product of ω's evaluated at different times and points and therefore, when averaged, will contain the second-order space-time correlation coefficient of the turbulent velocity field. Introducing this correlation coefficient:

$$R_2(\xi_1, \tau_1; \xi_2, \tau_2) \equiv \frac{\langle\omega(\xi_1, \tau_1)\omega(\xi_2, \tau_2)\rangle}{\langle\omega^2\rangle} = \langle\omega(\xi_1, \tau_1)\omega(\xi_2, \tau_2)\rangle,$$

we get from the expression above for σ_2,

$$\begin{aligned}\langle\sigma_2(\xi, \tau)\rangle = \int_0^\tau d\tau' \int_{-\infty}^{\infty} \frac{\partial}{\partial\xi} \Gamma(\xi - \xi', \tau - \tau')\, d\xi' \int_0^{\tau'} d\tau'' \\ \times \int_{-\infty}^{\infty} R_2(\xi', \tau'; \xi'', \tau'')\sigma_0(\xi'', \tau'') \frac{\partial}{\partial\xi} \Gamma(\xi' - \xi'', \tau' - \tau'')\, d\xi''.\end{aligned} \tag{17}$$

Similarly, the expression for σ_n involves a product of n ω's, each with a different argument, and so $\langle\sigma_n\rangle$ will involve the nth order correlation coefficient of the turbulent velocity field

$$R_n(\xi_1, \tau_1; \xi_2, \tau_2; \cdots; \xi_n, \tau_n) \equiv \langle\omega(\xi_1, \tau_1)\omega(\xi_2, \tau_2) \cdots \omega(\xi_n, \tau_n)\rangle.$$

Thus, when the power series representing $\langle\sigma\rangle$ is known to converge, a knowledge of all the correlation coefficients of the turbulent velocity field yields $\langle\sigma\rangle$ exactly as a function of ξ and τ, while if one knows the correlation coefficients R_n up through order n one has (in principle, assuming that all the integrations can be carried out) an nth order approximation to $\langle\sigma\rangle$.

5. Example: Dispersion at small times for initial distributions and space correlation coefficients of Gaussian type. We have seen that one situation in which the perturbation series (7) actually solves the initial value problem is when, for a given convection field (or class of convection fields), the time interval $0 \leq \tau \leq \tau_0$ is sufficiently small. Let us, then, calculate expressions for $\sigma_0(\xi, \tau)$ and $\langle\sigma_2(\xi, \tau)\rangle$ at small values of τ, for certain initial distributions and correlation functions. (We recall that the first-order term, $\langle\sigma_1\rangle$, vanishes.) In terms of the computation the restriction to small values of τ is imposed by the desire to approximate certain functions of τ so that they can be integrated (see Eq. (22) below).

Consider initial distributions of the form

$$\phi(\xi) = (4\pi\tau^*)^{-1/2} \exp(-\xi^2/4\tau^*),$$

depending on the parameter τ^*. Then from (10)

$$\sigma_0(\xi, \tau) = [4\pi(\tau + \tau^*)]^{-1/2} \exp[-\xi^2/4(\tau + \tau^*)]. \tag{18}$$

(This is the distribution that would result from molecular diffusion of a unit mass concentrated at $\xi = 0$, starting at time $\tau = -\tau^*$.)

We assume that the turbulence is homogeneous and stationary, and furthermore that the second-order correlation coefficient can be written as a function of the ξ's times a function of the τ's:

$$R_2(\xi', \tau'; \xi'', \tau'') = \mathcal{R}_2(\xi' - \xi''; \tau' - \tau'') = \mathcal{R}_\xi(\xi' - \xi'')\mathcal{R}_\tau(\tau' - \tau'').$$

Finally we consider the particular class of functions

$$\mathcal{R}_\xi(\xi) = \exp(-\xi^2/4b), \tag{19}$$

with the parameter b. Without specifying $\mathcal{R}_\tau$, we assume that in the neighborhood of $\tau = 0$ its rate of decrease is sufficiently slow so that for the small values of τ under consideration $\mathcal{R}_\tau(\tau) \approx 1$ (since $\mathcal{R}_\tau(0) = 1$). We shall, however, carry along the factor $\mathcal{R}_\tau$ in the calculations until it is necessary to invoke this assumption, so that perhaps some enterprising reader can find an explicit functional form for $\mathcal{R}_\tau$ which will obviate the approximations which are made at that point.

Under the foregoing assumptions, if we replace ξ'' and τ'' by the new variables of integration $\xi_2 = \xi' - \xi''$ and $\tau_2 = \tau' - \tau''$, Eq. (17) can be written

$$\begin{aligned}\langle\sigma_2(\xi, \tau)\rangle = &-\frac{1}{4\pi^{1/2}} \int_0^\tau \frac{d\tau'}{(\tau - \tau')^{3/2}} \int_{-\infty}^{\infty} (\xi' - \xi) \exp\left[-\frac{(\xi' - \xi)^2}{4(\tau - \tau')}\right] d\xi' \\ &\times \frac{1}{4\pi^{1/2}} \int_0^{\tau'} \frac{\mathcal{R}_\tau(\tau_2)\, d\tau_2}{\tau_2^{3/2}} \int_{-\infty}^{\infty} \xi_2 \mathcal{R}_\xi(\xi_2)\sigma_0(\xi' - \xi_2, \tau - \tau_2) \exp\left(-\frac{\xi_2^2}{4\tau_2}\right) d\xi_2,\end{aligned} \tag{20}$$

where $\partial\Gamma/\partial\xi$ has been obtained from (11). Inserting the expressions for σ_0 and $\mathcal{R}_\xi$ given by (18) and (19) into the first integral to be evaluated,

$$I_1^\xi(\xi', \tau', \tau_2) \equiv \int_{-\infty}^{\infty} \xi_2 \mathcal{R}_\xi(\xi_2)\sigma_0(\xi' - \xi_2, \tau' - \tau_2) \exp\left(-\frac{\xi_2^2}{4\tau_2}\right) d\xi_2,$$

we find, after an essentially simple integration, that

$$I_1^\xi = \frac{\xi' b^{3/2} \tau_2^{3/2}}{D^{3/2}} \exp\left[-\frac{\xi'^2(\tau_2 + b)}{4D}\right],$$

where

$$D = \tau_2(\tau' - \tau_2 + \tau^*) + b(\tau' + \tau^*). \tag{21}$$

Next, we must evaluate

$$\begin{aligned} I^\tau(\xi', \tau') &\equiv \frac{1}{4\pi^{1/2}} \int_0^{\tau'} \frac{\mathfrak{R}_\tau(\tau_2)}{\tau_2^{3/2}} I_1^\xi(\xi', \tau', \tau_2)\, d\tau_2 \\ &= \frac{\xi' b^{3/2}}{4\pi^{1/2}} \int_0^{\tau'} \frac{\mathfrak{R}_\tau(\tau_2)}{D^{3/2}} \exp\left[-\frac{\xi'^2(\tau_2 + b)}{4D}\right] d\tau_2\,, \end{aligned} \tag{22}$$

where D is given by (21). Since $\tau_2 \le \tau' \le \tau \ll 1$, $\mathfrak{R}_\tau(\tau_2) \approx 1$. Then, by considering different relative magnitudes for τ, τ^* and b, we can simplify D and evaluate I^τ in three cases: (1) $\tau \ll b$, $\tau^* \ll b$; (2) $\tau \ll b$, $\tau \ll \tau^*$; (3) $\tau \ll \tau^*$, $b \ll \tau^*$. In each case we shall evaluate $I^\tau(\xi', \tau')$, then (see Eq. (20))

$$I_2^\xi(\xi, \tau, \tau') \equiv \int_{-\infty}^{\infty} (\xi' - \xi) \exp\left[-\frac{(\xi' - \xi)^2}{4(\tau - \tau')}\right] I^\tau(\xi', \tau')\, d\xi', \tag{23}$$

and finally

$$\langle\sigma_2(\xi, \tau)\rangle \equiv -\frac{1}{4\pi^{1/2}} \int_0^{\tau} \frac{I_2^\xi(\xi, \tau, \tau')}{(\tau - \tau')^{3/2}}\, d\tau'. \tag{24}$$

Case 1: $\tau \ll b$, $\tau^* \ll b$. From (21), neglecting terms of second order in the small quantities τ, τ', τ_2 and τ^*, we get $D = b(\tau + \tau^*)$. We insert this expression for D and $\mathfrak{R}_\tau = 1$ in (22) to obtain

$$\begin{aligned} I^\tau &= \frac{\xi'}{4\pi^{1/2}} \int_0^{\tau'} \exp\left[-\frac{\xi'^2}{4(\tau' + \tau^*)}\right] \frac{d\tau_2}{(\tau' + \tau^*)^{3/2}} \\ &= \frac{\xi'\tau'}{4\pi^{1/2}(\tau' + \tau^*)^{3/2}} \exp\left[-\frac{\xi'^2}{4(\tau' + \tau^*)}\right]. \end{aligned}$$

Then, using this result in (23) and integrating gives

$$\begin{aligned} I_2^\xi &= \frac{\tau'}{4\pi^{1/2}(\tau' + \tau^*)^{3/2}} \int_{-\infty}^{\infty} \xi'(\xi' - \xi) \exp\left[-\frac{(\xi' - \xi)^2}{4(\tau - \tau')} - \frac{\xi'^2}{4(\tau' + \tau^*)}\right] d\xi' \\ &= -\frac{\tau'}{2} \frac{(\tau - \tau')^{3/2}}{(\tau + \tau^*)^{5/2}} [\xi^2 - 2(\tau + \tau^*)] \exp\left[-\frac{\xi^2}{4(\tau + \tau^*)}\right], \end{aligned}$$

and at last from (24) we get

$$\langle\sigma_2(\xi, \tau)\rangle = \frac{\tau^2}{16\pi^{1/2}} \frac{[\xi^2 - 2(\tau + \tau^*)]}{(\tau + \tau^*)^{5/2}} \exp\left[-\frac{\xi^2}{4(\tau + \tau^*)}\right]. \tag{25}$$

Case 2: $\tau \ll b$, $\tau \ll \tau^*$. Now $D = b\tau^*$ and

$$I^\tau = \frac{\xi'}{4\pi^{1/2}\tau^{*3/2}} \int_0^{\tau'} \exp\left(-\frac{\xi'^2}{4\tau^*}\right) d\tau_2\,.$$

Proceeding as in Case 1, we find

$$\langle \sigma_2(\xi, \tau) \rangle = \frac{\tau^2}{16\pi^{1/2}} \frac{(\xi^2 - 2\tau^*)}{\tau^{*5/2}} \exp\left(-\frac{\xi^2}{4\tau^*}\right). \tag{26}$$

Case 3: $\tau \ll \tau^*$, $b \ll \tau^*$. Now $D = \tau^*(\tau_2 + b)$. By performing the successive integrations indicated above (and using along the way the fact that $\tau' \ll \tau \ll \tau^*$), we obtain in this case

$$\langle \sigma_2(\xi, \tau) \rangle = \frac{\xi^2 - 2\tau^*}{4\pi^{1/2}\tau^{*5/2}} \exp\left(-\frac{\xi^2}{4\tau^*}\right)\left\{b\tau - 2b^{3/2}[(\tau + b)^{1/2} - b^{1/2}]\right\}. \tag{27}$$

Remarks. 1. The expressions for $\langle \sigma_2 \rangle$ in Cases 1 and 2, given by Eqs. (25) and (26) respectively, are quite similar in form; indeed, if in Case 1 we made the more restrictive assumption that $\tau \ll \tau^* \ll b$, (25) would reduce to (26). Equation (27), the result for Case 3, however, is different from the other two: only in this case does b appear in the expression for $\langle \sigma_2 \rangle$. Thus $\langle A_2 \rangle = \sigma_0 + \Omega^2 \langle \sigma_2 \rangle$, the second-order approximation to $\langle \sigma_2 \rangle$, does not depend on b when $\tau \ll b$, whereas in the absence of this assumption the expression for $\langle A_2 \rangle$ does involve b. (It should be noted that b is proportional to L^2, where L is the length scale of turbulence, the integral of the correlation coefficient from 0 to ∞ with respect to x.)

2. If in Case 3 the more restrictive assumption that $\tau \ll b \ll \tau^*$ is made, we get $\langle \sigma_2 \rangle = 0$, for

$$(\tau + b)^{1/2} - b^{1/2} = b^{1/2}\left[\left(1 + \frac{\tau}{b}\right)^{1/2} - 1\right] \approx b^{1/2}\left[\left(1 - \frac{\tau}{2b}\right) - 1\right] = \frac{\tau}{2b^{1/2}} \cdot$$

3. A comparison of Eqs. (18) and (25) shows that in Case 2 $\langle \sigma_2 \rangle = (\tau^2/2)\partial\sigma_0/\partial\tau$. It is not clear what significance, if any, there is in this interesting relation.

4. Within the limitations of requirement (15), our sufficient condition for the validity of the perturbation scheme, it can be shown that in the region of the ξ, τ-plane where

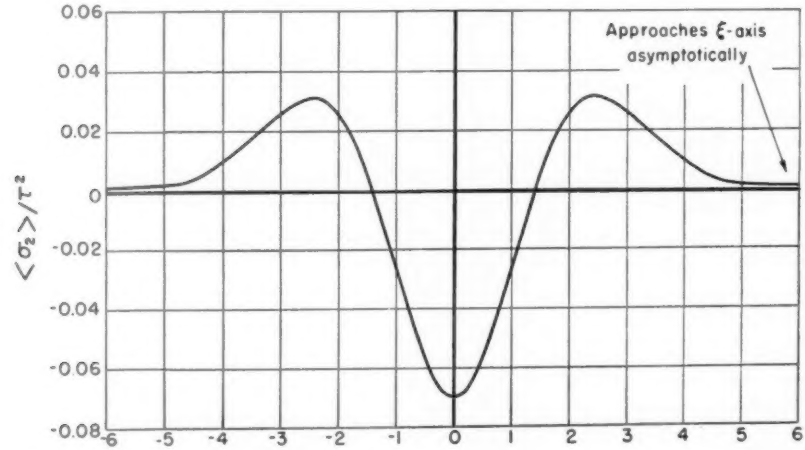

FIG. 1. Example of second-order correction to mean concentration at small times: $\langle \sigma_2 \rangle / \tau^2$ plotted as a function of ξ from Eq. (26) with $\tau^* = 1$.

$\langle A_2\rangle$ is a good approximation to $\langle\sigma\rangle$, $\Omega^2\langle\sigma_2\rangle$ is much smaller than σ_0. (This is not surprising since in perturbing with respect to the convection term, we assumed this term is small in comparison with the others in the original differential equation.) It should be kept in mind, of course, that sufficient conditions for a certain result flow out of the particular mathematical technique employed, and often the result is valid under much broader conditions. This is particularly true of perturbation methods. Nevertheless, even if we limit ourselves to values of the parameters and variables which satisfy (15), the results are of interest. Equations (25), (26) and (27) show that in all three cases, for given values of b and τ^* and a fixed value of τ, $\langle\sigma_2\rangle$ is negative at $\xi^2 = 0$, increases, with increasing ξ^2, to a positive maximum, and then approaches 0 asymptotically as $\xi^2 \to \infty$. This is illustrated in the figure for Case 2. The exponential factors in both σ_0 and $\langle\sigma_2\rangle$ make them go to 0 as $\xi^2 \to \infty$, but the *relative* magnitude of $\langle\sigma_2\rangle$, as compared with σ_0, increases with ξ^2; indeed, $\langle\sigma_2\rangle/\sigma_0$ becomes proportional to ξ^2 as ξ^2 increases. Adding turbulence to molecular diffusion, then, "sweeps out" the initial distribution more rapidly, and this additional dispersive effect assumes a particular functional form, at least in this set of examples.

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WAVE PROPAGATION IN RODS OF VOIGT MATERIAL AND VISCO-ELASTIC MATERIALS WITH THREE-PARAMETER MODELS*

BY

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Abstract. The problem of an impulsively applied and subsequently maintained constant velocity at the end of a semi-infinite rod of Voigt material is considered and integral expressions are obtained for the velocity and stress distributions in the rod. Apart from a constant factor the stress distribution arising from an impulsively applied constant stress at the end of the rod is the same as the velocity solution above. The same problem is considered also for materials with three-parameter models. The stress distributions for both the case of constant applied stress and constant applied velocity are represented graphically for the materials considered, dimensionless co-ordinates being used.

1. Fundamental considerations and the Voigt model. Lee and Kanter[1] considered the problem of an impulsively applied constant velocity on the end of a rod of Maxwell material and analytically determined the subsequent stress distribution in the rod. Glauz and Lee[2] treated the same problem for a four-parameter model visco-elastic material and determined the stress and velocity distributions by the method of characteristics. In order to complete the picture we consider here the other two-parameter model, which corresponds to the Voigt material, and also the three-parameter models.

The Voigt model is shown in Fig. 1 and is composed of a spring and a dashpot in

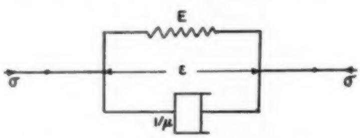

FIG. 1. Model for Voigt material.

parallel and so the Voigt material presents a retarded elastic response to stress. The stress strain relation is

$$\sigma = E\epsilon + \frac{1}{\mu}\epsilon_t , \tag{1}$$

where σ is the stress, ϵ the strain, the suffix denotes differentiation with respect to time and E and μ are elastic and viscous constants, respectively, for the material.

Let us consider a creep test, together with unloading, in which a constant stress σ_0 is suddenly applied at time $t = 0$ and suddenly removed at $t = t_0$ so that $\sigma' = \sigma_0\{H(t) - H(t - t_0)\}$ where $H(t)$ is Heavisides' step function,

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$$

*Received June 10, 1955. Sponsored under a Department of Defense Contract, NOrd-11496 between the Bureau of Ordnance, Navy Department, and Brown University.

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From Eq. (1) it is found by integration that $\epsilon = \{\psi(t)H(t) - \psi(t - t_0)H(t - t_0)\}$, where $\psi(t) = (\sigma_0/E)(1 - e^{-E\mu t})$. For convenience we introduce dimensionless variables $\tau = E\mu t$, $\Sigma' = \sigma'/\sigma_0$ and $\mathcal{E} = (E\epsilon)/\sigma_0$. Then,

$$\Sigma' = \{H(\tau) - H(\tau - \tau_0)\},$$
$$\mathcal{E} = \{\varphi(\tau)H(\tau) - \varphi(\tau - \tau_0)H(\tau - \tau_0)\}, \tag{2}$$

where

$$\varphi(\tau) = (1 - e^{-\tau}). \tag{3}$$

The creep test is represented in Fig. 2 where $\tau_0 = E\mu t_0$ has been given the value 2.

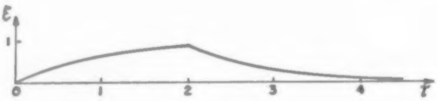

FIG. 2.

We now turn our attention to the problem at hand. We are concerned with the propagation of longitudinal waves in a semi-infinite rod, $x \geq 0$, which is initially unstrained and at rest. Let $u(x, t)$ denote the displacement of the section x of the rod so that its position at time t is given by $(x + u)$. Then, if ρ is the density of the unstrained material,

$$\rho u_{tt} = -\sigma_x , \tag{4}$$

$$\epsilon = -u_x , \tag{5}$$

where suffixes denote partial differentiation with respect to the corresponding variable. Here σ and ϵ are the nominal compressive stress and nominal compressive strain, respectively and it is supposed that the stress strain relation of Eq. (1) applies to these nominal values.

Eliminating σ and ϵ from Eqs. (1), (4) and (5) we obtain

$$\rho u_{tt} = Eu_{xx} + \frac{1}{\mu} u_{xxt} , \tag{6}$$

from which it is readily verified that the stress σ and the particle velocity $v = u_t$ both satisfy the partial differential equation

$$\rho f_{tt} = Ef_{xx} + \frac{1}{\mu} f_{xxt} . \tag{7}$$

It follows that the stress distribution $\sigma'(x, t)$ in the case in which a constant stress σ_0 is suddenly applied at the end of the rod and then maintained is of the same form as the velocity distribution $v(x, t)$ for the case of a suddenly applied and subsequently maintained velocity v_0 at the end of the rod. In fact,

$$\frac{v(x, t)}{v_0} = \frac{\sigma'(x, t)}{\sigma_0} . \tag{8}$$

If we introduce the dimensionless variables $\xi = (\rho E)^{1/2}\mu x$, $V = v(x, t)/v_0$ and, as before, $\tau = E\mu t$, $\Sigma' = \sigma'(x, t)/\sigma_0$, Eq. (8) becomes

$$\Sigma'(\xi, \tau) = V(\xi, \tau). \tag{9}$$

We therefore confine our attention to the problem in which the end of the rod (which is initially unstrained and at rest) is suddenly given a velocity v_0 which is subsequently maintained and determine the stress and velocity solutions, $\sigma(x, t)$ and $v(x, t)$. We introduce the dimensionless stress $\Sigma(\xi, \tau) = \sigma(x, t)/(\rho E)^{1/2}v_0$. We will then have determined the stress solutions $\Sigma(\xi, \tau)$ and $\Sigma'(\xi, \tau)$ both for the case of constant applied velocity and constant applied stress.

The determination of the solutions, for the Voigt material, is carried out in Appendix A by the method of Laplace transforms. Zverev[3] adopted the same method but carried out the inversion of the Laplace transform in a different manner and obtained an infinite integral expression for the velocity distribution. The present author gave a similar expression for the stress distribution[4] but the method adopted in this paper is readily extended to the case of the three-parameter model with two viscous elements and in addition the finite integral expressions obtained are readily computed for the moderate values of τ in which we are interested.

The stress solutions $\Sigma(\xi, \tau)$ and $\Sigma'(\xi, \tau)$ as given by Eqs. (A.18) and (A.14), respectively, are depicted graphically in Figs. 3(a) and 3(b). In Fig. 3(a), Σ is plotted against

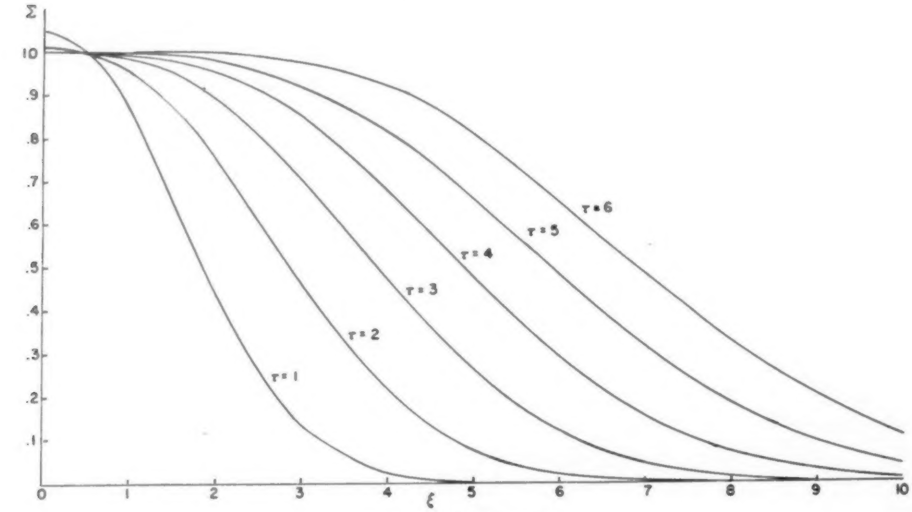

FIG. 3(a). Voigt model.

ξ for different values of τ so that the distribution of stress in the rod at various times, arising from an impulsively applied and subsequently maintained constant velocity at the end of the rod, is illustrated. From Eq. (A.19) it is seen that the stress at the end of the rod, which is very large immediately after impact, decreases rapidly at first and subsequently continues to decrease, but with diminishing rapidity, ultimately approaching the value unity. The stress at a position of the rod very close to the end rises extremely

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rapidly shortly after impact to a large value and then commences to decrease. The occurrence of large stresses near the end of the rod is due to the response of the viscous element to an attempted sudden change in strain. Such a change in strain would correspond to an infinite strain rate and so demand an infinite stress to produce it.

However, as the time increases the viscous behavior tends to die out and the rod tends to behave as in the case of a completely elastic material. At each position in the rod the stress ultimately approaches the value unity. It is noticed that there is a considerable spread in the stress distribution, as opposed to the sharp wave-front occurring in the case of an elastic material, and that the effect of the disturbance is propagated instantaneously (in a decaying manner) to all positions in the rod.

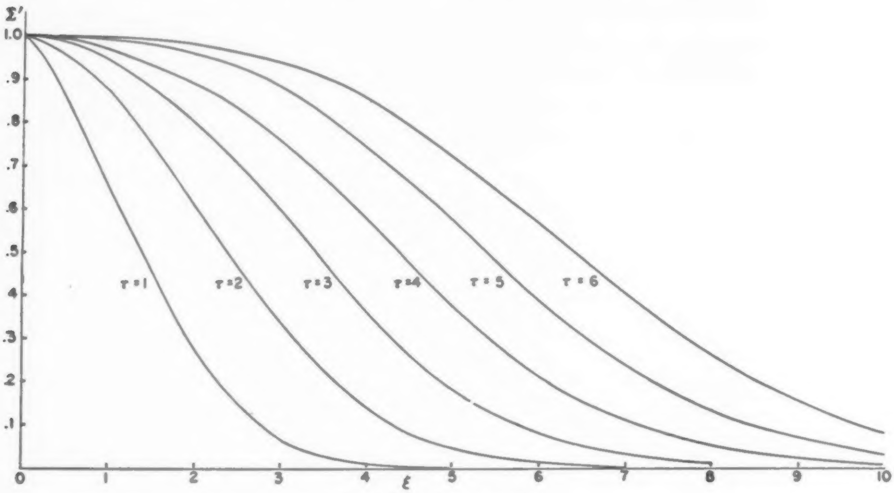

FIG. 3(b). Voigt model.

In Fig. 3(b), Σ' is plotted against ξ for different values of τ so that the distribution of stress in the rod at various times, arising from an impulsively applied and subsequently maintained constant stress at the end of the rod, is illustrated. At each position in the rod Σ' ultimately approaches the value unity but it is again interesting to note the considerable spread in the stress distribution.

2. The three-parameter model with one elastic and two viscous elements. There are four true three-parameter models, all the others reducing effectively to models with fewer parameters. We first consider the two models shown in Figs. 4(a) and 4(b) which

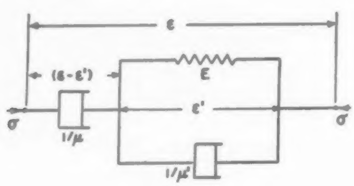

FIG. 4(a).

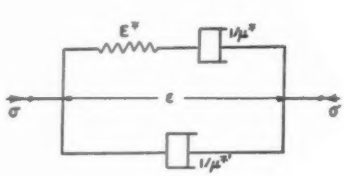

FIG. 4(b).

have two dashpots and one spring. For the model of Fig. 4(a) we have

$$E\epsilon' + \frac{1}{\mu'}\epsilon'_t = \sigma = \frac{1}{\mu}(\epsilon_t - \epsilon'_t). \tag{10}$$

Hence, eliminating ϵ',

$$E\epsilon_t + \frac{\epsilon_{tt}}{\mu'} = E\mu\sigma + \left(1 + \frac{\mu}{\mu'}\right)\sigma_t\,. \tag{11}$$

In a similar manner we obtain the stress strain relation for the model of Fig. 4(b),

$$\frac{\sigma_t}{E^*} + \mu^*\sigma = \epsilon_t\left(1 + \frac{\mu^*}{\mu^{*\prime}}\right) + \frac{\epsilon_{tt}}{\mu^{*\prime}E^*}\,. \tag{12}$$

If we set $(E/E^*)^{1/2} = \mu^*/\mu = \mu^{*\prime}/\mu' = (1 + \mu/\mu')$, then Eqs. (11) and (12) become identical. Consequently we confine our attention to the model of Fig. 4(a). If we set $\sigma = \sigma_0 H(t)$ in Eq. (11) and integrate we find that

$$\epsilon = \frac{\sigma_0}{E}\,[E\mu t + (1 - e^{-E\mu' t})]H(t), \tag{13}$$

so that the creep test with unloading is given by Eq. (2) with

$$\varphi(\tau) = \{\tau + (1 - e^{-\tau/\lambda})\}, \tag{14}$$

where $\lambda = \mu/\mu'$. This is depicted graphically in Fig. 5 for the particular case $\lambda = 1$.

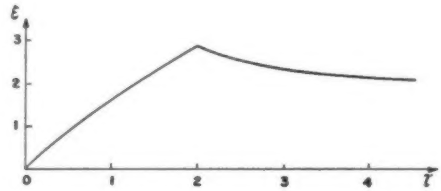

FIG. 5.

It is noted that after the removal of the applied stress the strain approaches an asymptotic value given by $\epsilon = \tau_0$ $(= 2)$ and this residual strain is due to the viscous flow in the dashpot with viscous constant μ which is in series with the retarded elastic or Voigt element. The strain of the delayed elastic portion is asymptotically recovered, as is seen from Fig. 2.

Eliminating σ and ϵ from Eqs. (4), (5) and (11) and setting $v = u_t$,

$$\rho E\mu v_t + \rho\left(1 + \frac{\mu}{\mu'}\right)v_{tt} = Ev_{xx} + \frac{1}{\mu'}v_{xxt}\,, \tag{15}$$

and it is readily verified that σ satisfies the same equation. Hence Eq. (9) again holds and the stress distributions $\Sigma(\xi, \tau)$ and $\Sigma'(\xi, \tau)$ arising when the end of a rod, initially unstrained and at rest, of a material corresponding to the three-parameter model of Fig. 4(a) is impulsively subjected to constant applied velocity and constant applied stress, respectively, are determined in Appendix B, Eqs. (B.8) and (B.14).

Σ and Σ' are plotted in Figs. 6(a) and 6(b), respectively, against ξ for different values of τ in the particular case $\lambda = 1$, so that the two viscous constants in the model are equal. Thus in Fig. 6(a) the stress distribution in the rod at various times, arising from an impulsively applied and subsequently maintained constant velocity at the end of the rod, is illustrated. The stress at the end of the rod is very large immediately after impact due to the combined "rigid" responses of the viscous element and delayed elastic element, in series, which comprise the model of Fig. 4(a). Nevertheless it decreases very rapidly and has a value less than unity at time $\tau = 1$. The stress at the end of the rod continues

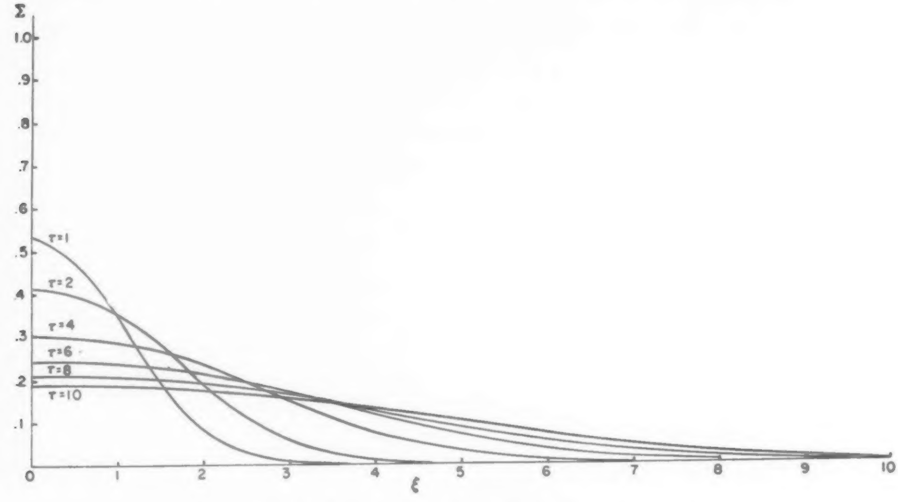

FIG. 6(a). Three parameter (1 elastic, 2 viscous).

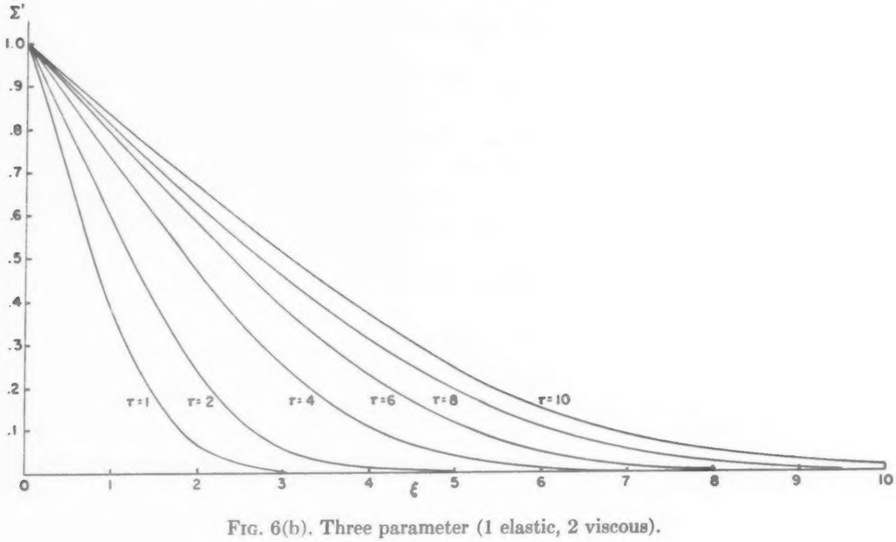

FIG. 6(b). Three parameter (1 elastic, 2 viscous).

to decrease as τ increases and at each position in the rod the stress ultimately vanishes.

In Fig. 6(b) the stress distribution in the rod at various times, arising from an impulsively applied and subsequently maintained constant stress at the end of the rod, is illustrated. At each position in the rod Σ' ultimately approaches the value unity.

3. The three-parameter model with one viscous and two elastic elements. We now consider the remaining two true three-parameter models which are shown in Figs. 7(a) and 7(b), these having two springs and one dashpot.

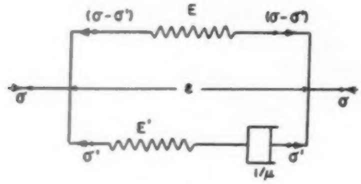

FIG. 7(a).

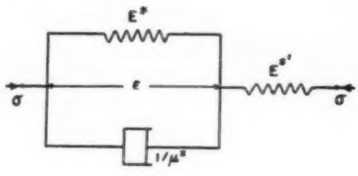

FIG. 7(b).

For the model of Fig. 7(a),

$$\epsilon_t = \frac{\sigma_t'}{E'} + \mu\sigma',$$
$$\epsilon = \frac{1}{E}(\sigma - \sigma'). \tag{16}$$

Hence, eliminating σ',

$$\frac{\sigma_t}{E'} + \mu\sigma = \epsilon_t\left(1 + \frac{E}{E'}\right) + E\mu\epsilon. \tag{17}$$

In a similar manner we obtain the stress strain relation for the model of Fig. 7(b),

$$E^*\epsilon + \frac{1}{\mu^*}\epsilon_t = \sigma\left(1 + \frac{E^*}{E^{*\prime}}\right) + \frac{\sigma_t}{E^{*\prime}\mu^*}\,. \tag{18}$$

If we let $E^*/E = E^{*\prime}/E' = (\mu/\mu^*)^{1/2} = 1 + (E/E')$, then Eqs. (17) and (18) become identical so that we confine our attention to the model of Fig. 7(a). If we set $\sigma = \sigma_0 H(t)$ in Eq. (17) and integrate we find that

$$\epsilon = \frac{\sigma_0}{E}\left\{1 - \frac{E'}{(E + E')}\exp\left[-\frac{EE'\mu t}{(E + E')}\right]\right\}H(t), \tag{19}$$

so that the creep test with unloading is give by Eq. (2) with

$$\varphi(\tau) = \left\{1 - \frac{1}{(1 + k)}\exp\left[\frac{-\tau}{(1 + k)}\right]\right\}, \tag{20}$$

where $k = E/E'$. This is depicted graphically in Fig. 8 for the particular case $k = 1$.

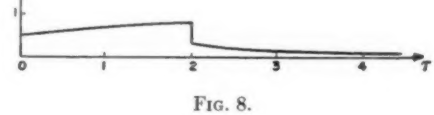

FIG. 8.

For this model there is an instantaneous elastic strain of magnitude $\sigma_0/(E + E') = \sigma_0/E^{*\prime}$ when the stress is applied and the strain instantaneously falls by this same amount when the applied stress is removed, and asymptotically tends to zero. (This behavior is perhaps more apparent from Fig. 7(b), in which there is a spring of elastic constant $E^{*\prime}$ in series with a retarded elastic element.)

Eliminating σ and ϵ from Eqs. (4), (5) and (17),

$$\frac{\rho u_{ttt}}{E'} + \rho\mu u_{tt} = u_{xxt}\left(1 + \frac{E}{E'}\right) + E\mu u_{xx} , \tag{21}$$

and it may be verified that σ and v satisfy the same equation. The stress distributions $\Sigma(\xi, \tau)$ and $\Sigma'(\xi, \tau)$ for the problems previously considered for the other materials are determined for this material in Appendix C, Eqs. (C.20) and (C.19), respectively. From the prescribed condition on the end of the rod $\Sigma'(0, \tau) = 1$ and a simpler expression for $\Sigma(0, \tau)$ is given in Eq. (C.25), so that these provide a check on the accuracy of the computation of $\Sigma'(\xi, \tau)$ and $\Sigma(\xi, \tau)$ from Eqs. (C.19) and (C.20).

Σ and Σ' are plotted in Figs. 9(a) and 9(b), respectively, against ξ for different values

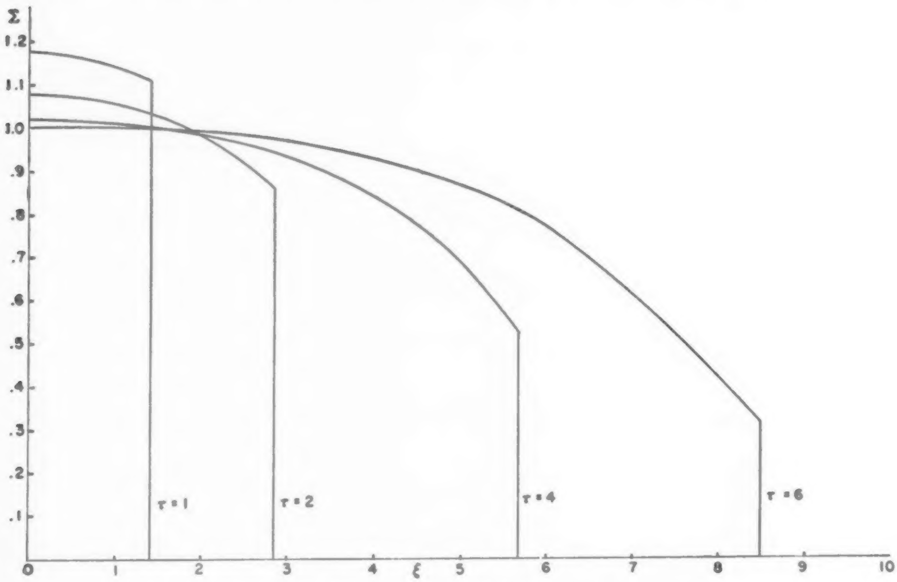

Fig. 9(a). Three parameter (2 elastic, 1 viscous).

of τ in the particular case $k = 1$, so that the two elastic constants in the model are equal. Thus in Fig. 9(a) the stress distribution in the rod at various times, arising from an impulsively applied and subsequently maintained constant velocity at the end of the rod, is illustrated. A wave of stress discontinuity is propagated along the rod with constant velocity $(2E/\rho)^{1/2}$ (or in the general case $[(1 + k)E/k\rho]^{1/2} = [(E + E')/\rho]^{1/2}$), relative to the unstrained rod. Hence at time τ the portion of the rod for which $\xi > (2)^{1/2}\tau$ (or $\xi > (k + 1)^{1/2}\tau/k^{1/2}$, in the general case) is unstressed. The magnitude of the stress discontinuity at $\xi = (2)^{1/2}\tau$, (or $\xi = (k + 1)^{1/2}\tau/k^{1/2}$), is $(2)^{1/2}$ exp $(-\tau/4)$, (or $[(k +$

$1)/k]^{1/2}$ exp $[-\tau/2k(k + 1)]$), so that it decreases exponentially as the wavefront advances along the rod. In the case considered, the stress at the end of the rod jumps to the value $(2)^{1/2}$ immediately after impact and then monotonically decreases, ultimately approaching the value unity. The stress at each position in the rod also ultimately approaches the value unity.

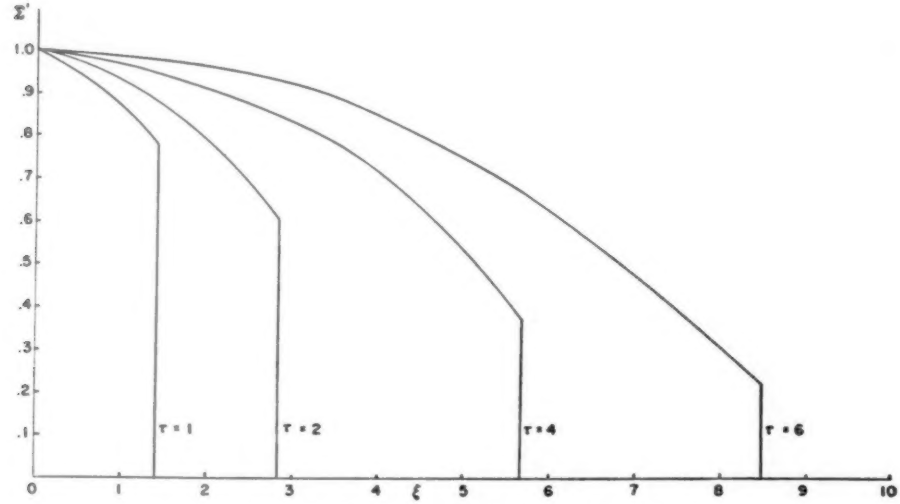

FIG. 9(b). Three parameter (2 elastic, 1 viscous).

In Fig. 9(b) the stress distribution in the rod at various times, due to an impulsively applied and subsequently maintained constant stress at the end of the rod, is illustrated. Here again a wave of stress discontinuity is propagated along the rod, the constant velocity of propagation being the same as before. In this case the magnitude of the stress discontinuity at $\xi = (2)^{1/2}\tau$, (or at $\xi = (k + 1)^{1/2}\tau/k^{1/2}$ in the general case), is exp $(-\tau/4)$, (or exp $[-\tau/2k(k + 1]$), so that again it decreases exponentially as the wavefront advances along the rod. The stress at each position in the rod ultimately approaches the value unity.

4. Concluding remarks. We have been concerned with the stress distributions in rods, of Voigt material and of visco-elastic materials with three-parameter models, impulsively subjected to constant applied velocity or constant applied stress at the end of the rod. We have seen that there are two basic types of true three-parameter models, namely those with one spring and two dashpots and those with one dashpot and two springs. In a subsequent paper it is intended to compare the stress distributions $\Sigma(\xi, \tau)$ and $\Sigma'(\xi, \tau)$ for seven different visco-elastic materials; elastic, viscous, Maxwell, Voigt, the two basic three-parameter models considered here and also the four-parameter model considered by Glauz and Lee[2]. With this in mind we chose the model of Fig. 4(a), in preference to that of Fig. 4(b), since the ultimate behavior may then be compared with that of a dashpot with viscous constant μ. Similarly we chose the model of Fig. 7(a) since its ultimate behavior may be compared with that of the Voigt model considered in Sec. 1.

Acknowledgement. The author wishes to express his gratitude to Professor E. H. Lee for his advice and encouragement during the preparation of this work, and also to various members of the computing staff at Brown University for carrying out the numerical integrations.

Appendix A. From Eq. (7) it is found that $V(\xi, \tau)$ satisfies the equation

$$V_{\tau\tau} = V_{\xi\xi} + V_{\xi\xi\tau} . \tag{A.1}$$

Also, from Eq. (4),

$$V_\tau = -\Sigma_\xi . \tag{A.2}$$

In addition there are the boundary conditions,

$$\begin{gathered} V(\infty, \tau) = 0 = \Sigma(\infty, \tau), \\ V(0, \tau) = 1. \end{gathered} \tag{A.3}$$

Applying the Laplace transform,

$$L\{f(\xi, \tau)\} = \int_0^\infty e^{-q\tau} f(\xi, \tau)\, d\tau, \tag{A.4}$$

to Eqs. (A.1-A.3) and remembering that the rod is initially unstressed, unstrained and at rest, we obtain

$$q^2 L\{V\} = (1 + q)[L\{V\}]_{\xi\xi} = (1 + q)L\{V_{\xi\xi}\} \tag{A.5}$$

$$qL\{V\} = -[L\{\Sigma\}]_\xi = -L\{\Sigma_\xi\} \tag{A.6}$$

and

$$\begin{gathered} L\{V(\infty, \tau)\} = 0 = L\{\Sigma(\infty, \tau)\}, \\ L\{V(0, \tau)\} = \frac{1}{q} \cdot \end{gathered} \tag{A.7}$$

Solving the ordinary differential Equation (A.5) for $L\{V(\xi, \tau)\}$ and applying the boundary conditions (A.7), we obtain

$$L\{V(\xi, \tau)\} = \frac{1}{q} \exp\left[\frac{-\xi q}{(1 + q)^{1/2}}\right]. \tag{A.8}$$

From Eq. (A.6) and the boundary condition in (A.7),

$$L\{\Sigma(\xi, \tau)\} = \frac{(1 + q)^{1/2}}{q} \exp\left[\frac{-\xi q}{(1 + q)^{1/2}}\right]. \tag{A.9}$$

In carrying out the inversions of $L\{V\}$ and $L\{\Sigma\}$ we make use of the following general result for Laplace transforms which is valid under certain conditions.[5]

If

$$F(q) = L\{f(\tau)\} \qquad \text{and} \qquad \frac{1}{g(q)} \exp\left[-\eta h(q)\right] = L\{\varphi(\tau, \eta)\}$$

[5] H. S. Carslaw and J. C. Jaeger, *Operational methods in applied mathematics*, Oxford University Press, London, 1947, p. 259.

then

$$\frac{1}{g(q)} F[h(q)] = L\left\{\int_0^\infty \varphi(\tau, \eta) f(\eta)\, d\eta\right\}. \tag{A.10}$$

Now[6],

$$\frac{1}{(q)^{1/2}} \exp\,[-\xi(q)^{1/2}] = L\left\{\frac{\exp\,(-\xi^2/4\tau)}{(\pi\tau)^{1/2}}\right\}, \tag{A.11}$$

and

$$\frac{1}{(q)^{1/2}} \exp\left(-\frac{\eta}{q}\right) = L\left\{\frac{\cos\,[2(\eta\tau)^{1/2}]}{(\pi\tau)^{1/2}}\right\}$$

so that

$$\frac{1}{(q)^{1/2}} \exp\left[-\eta\left(q + \frac{1}{q} - 2\right)\right] = L\left\{\frac{\cos\,[2(\eta)^{1/2}(\tau - \eta)^{1/2}]}{(\pi)^{1/2}(\tau - \eta)^{1/2}}\, e^{2\eta} H(\tau - \eta)\right\}. \tag{A.12}$$

Hence, from the result of (A.10), for $q > 1$,

$$\frac{1}{(q-1)} \exp\left[\frac{-\xi(q-1)}{(q)^{1/2}}\right] = L\left\{\frac{1}{\pi}\int_0^\tau \frac{\cos\,[2(\eta)^{1/2}(\tau - \eta)^{1/2}]}{(\eta)^{1/2}(\tau - \eta)^{1/2}} \exp\left(2\eta - \frac{\xi^2}{4\eta}\right) d\eta\right\}. \tag{A.13}$$

Applying the shift theorem we obtain from Eq. (A.8),

$$\Sigma'(\xi, \tau) = V(\xi, \tau) = \frac{e^{-\tau}}{\pi}\int_0^\tau \frac{\cos\,[2(\eta)^{1/2}(\tau - \eta)^{1/2}]}{(\eta)^{1/2}(\tau - \eta)^{1/2}} \exp\left(2\eta - \frac{\xi^2}{4\eta}\right) d\eta. \tag{A.14}$$

Further,

$$\exp\left(-\frac{\eta}{q}\right) = L\left\{\delta(\tau) - \left(\frac{\eta}{\tau}\right)^{1/2} J_1[2(\eta\tau)^{1/2}]\right\}, \tag{A.15}$$

where $\delta(\tau)$ is the delta function. Thus,

$$\exp\left[-\eta\left(q + \frac{1}{q} - 2\right)\right] = L\left\{e^{2\eta}\left[\delta(\tau - \eta) - \frac{(\eta)^{1/2}}{(\tau - \eta)^{1/2}} J_1\{2(\eta)^{1/2}(\tau - \eta)^{1/2}\} \cdot H(\tau - \eta)\right]\right\}. \tag{A.16}$$

From Eqs. (A.10) and (A.11) it then follows that, for $q > 1$,

$$\frac{(q)^{1/2}}{(q-1)} \exp\left[\frac{-\xi(q-1)}{(q)^{1/2}}\right] = L\left\{\frac{\exp\,(2\tau - \xi^2/4\tau)}{(\pi\tau)^{1/2}} - \int_0^\tau \frac{J_1[2(\eta)^{1/2}(\tau - \eta)^{1/2}]}{(\pi)^{1/2}(\tau - \eta)^{1/2}} \exp\left(2\eta - \frac{\xi^2}{4\eta}\right) d\eta\right\}. \tag{A.17}$$

[6]R. V. Churchill, *Modern operational mathematics in engineering*, McGraw-Hill Book Co., Inc., New York, 1944, p. 299.

Applying the shift theorem we obtain from Eq. (A.9),

$$\Sigma(\xi, \tau) = \frac{e^{\tau}}{(\pi)^{1/2}} \left\{ \frac{\exp(-\xi^2/4\tau)}{(\tau)^{1/2}} - \int_0^{\tau} \frac{J_1[2(\eta)^{1/2}(\tau-\eta)^{1/2}]}{(\tau-\eta)^{1/2}} \exp\left[-2(\tau-\eta) - \frac{\xi^2}{4\eta}\right] d\eta \right\}. \tag{A.18}$$

Also, since

$$L\{\Sigma(0, \tau)\} = \frac{(1+q)^{1/2}}{q} = \frac{(q+1)^{1/2}}{[(q+1)-1]},$$

we find that

$$\Sigma(0, \tau) = \left\{ \frac{e^{-\tau}}{(\pi\tau)^{1/2}} + \operatorname{erf}[(\tau)^{1/2}] \right\}, \tag{A.19}$$

and

$$\frac{d}{d\tau}\{\Sigma(0, \tau)\} = \frac{-e^{-\tau}}{2(\pi)^{1/2}(\tau)^{3/2}}.$$

Appendix B. Equation (15) in dimensionless form becomes

$$V_\tau + (1 + \lambda)V_{\tau\tau} = V_{\xi\xi} + \lambda V_{\xi\xi\tau}. \tag{B.1}$$

Equations (A.2) and (A.3) still hold and applying the Laplace transform we obtain

$$q[1 + (1+\lambda)q]L\{V\} = (1 + \lambda q)L\{V_{\xi\xi}\}, \tag{B.2}$$

together with Eqs. (A.6) and (A.7). Solving Eq. (B.2) for $L\{V(\xi, \tau)\}$ and applying the boundary conditions of (A.7),

$$L\{V(\xi, \tau)\} = \frac{1}{q} \exp\left\{ -\frac{\xi q[1 + (1+\lambda)q]^{1/2}}{(1+\lambda q)^{1/2}} \right\}. \tag{B.3}$$

From Eq. (A.6) and the boundary condition in (A.7),

$$L\{\Sigma(\xi, \tau)\} = \frac{(1+\lambda q)^{1/2}}{(q)^{1/2}[1 + (1+\lambda)q]^{1/2}} \exp\left\{ -\frac{\xi q[1 + (1+\lambda)q]^{1/2}}{(1+\lambda q)^{1/2}} \right\}. \tag{B.4}$$

The inversion of $L\{\Sigma(\xi, \tau)\}$ will be carried out in a manner similar to that of Appendix A.

$$\exp\left[-\eta\left\{ q + \frac{1}{\lambda^2(1+\lambda)q} - \frac{(2+\lambda)}{\lambda(1+\lambda)} \right\} \right] = L\{\psi(\tau, \eta)\}$$

where

$$\psi(\tau, \eta) = \left\{ \exp\left[\frac{(2+\lambda)\eta}{\lambda(1+\lambda)}\right] \right\} \left[\delta(\tau-\eta) - \frac{(\eta)^{1/2}}{\lambda(1+\lambda)^{1/2}(\tau-\eta)^{1/2}} J_1\left\{ \frac{2(\eta)^{1/2}(\tau-\eta)^{1/2}}{\lambda(1+\lambda)^{1/2}} \right\} H(\tau-\eta) \right] \Bigg\}. \tag{B.5}$$

Hence, using the shift theorem,

$$\exp\left\{-\frac{\eta\lambda q[1+(1+\lambda)q]}{(1+\lambda)(1+\lambda q)}\right\} = L\{e^{-\tau/\lambda}\psi(\tau,\eta)\}. \tag{B.6}$$

But,

$$\frac{1}{(q)^{1/2}}\exp\left[-\frac{\xi(1+\lambda)^{1/2}(q)^{1/2}}{(\lambda)^{1/2}}\right] = L\left\{\frac{1}{(\pi\tau)^{1/2}}\exp\left[-\frac{(1+\lambda)\xi^2}{4\lambda\tau}\right]\right\}. \tag{B.7}$$

Hence, from the result of (A.10), we obtain

$$\Sigma(\xi,\tau) = \frac{1}{(\lambda)^{1/2}(1+\lambda)(\pi)^{1/2}}\exp\left[\frac{\tau}{\lambda(1+\lambda)}\right]\left\{\frac{\lambda(1+\lambda)^{1/2}}{(\tau)^{1/2}}\exp\left[-\frac{(1+\lambda)\xi^2}{4\lambda\tau}\right]\right.$$

$$\left.-\int_0^\tau J_1\left[\frac{2(\eta)^{1/2}(\tau-\eta)^{1/2}}{\lambda(1+\lambda)^{1/2}}\right]\exp\left[-\frac{(2+\lambda)(\tau-\eta)}{\lambda(1+\lambda)}-\frac{(1+\lambda)\xi^2}{4\lambda\eta}\right]\frac{d\eta}{(\tau-\eta)^{1/2}}\right\}. \tag{B.8}$$

We now carry out the inversion of $L\{V(\xi,\tau)\}$.

$$\frac{1}{q}\exp\left[-\eta\left\{q+\frac{1}{\lambda^2(1+\lambda)q}-\frac{(2+\lambda)}{\lambda(1+\lambda)}\right\}\right] = L\{\chi(\tau,\eta)\},$$

where

$$\chi(\tau,\eta) = \exp\left[\frac{\eta(2+\lambda)}{\lambda(1+\lambda)}\right]J_0\left[\frac{2(\eta)^{1/2}(\tau-\eta)^{1/2}}{\lambda(1+\lambda)^{1/2}}\right]H(\tau-\eta). \tag{B.9}$$

Hence, using the shift theorem,

$$\frac{\lambda}{(1+\lambda q)}\exp\left\{\frac{-\eta\lambda q[1+(1+\lambda)q]}{(1+\lambda)(1+\lambda q)}\right\} = L\{e^{-\tau/\lambda}\chi(\tau,\eta)\}. \tag{B.10}$$

From (B.6) and (B.10) we find that

$$\frac{\lambda^2[1+(1+\lambda)q]}{(1+\lambda q)}\exp\left\{\frac{-\eta\lambda q[1+(1+\lambda)q]}{(1+\lambda)(1+\lambda q)}\right\} = L\{e^{-\tau/\lambda}[\lambda(1+\lambda)\psi(\tau,\eta)$$

$$-\chi(\tau,\eta)]\}. \tag{B.11}$$

But,

$$\frac{1}{q}\exp\left[\frac{-\xi(1+\lambda)^{1/2}(q)^{1/2}}{(\lambda)^{1/2}}\right] = L\left\{\operatorname{erfc}\left[\frac{\xi(1+\lambda)^{1/2}}{2(\lambda\tau)^{1/2}}\right]\right\}. \tag{B.12}$$

Hence, from (B.11) and (B.12), using the result of (A.10),

$$\Sigma'(\xi,\tau) = V(\xi,\tau) = e^{-\tau/\lambda}\int_0^\infty\left[\psi(\tau,\eta)-\frac{\chi(\tau,\eta)}{\lambda(1+\lambda)}\right]\operatorname{erfc}\left[\frac{\xi(1+\lambda)^{1/2}}{2(\lambda\eta)^{1/2}}\right]d\eta. \tag{B.13}$$

Thus, finally,

$$\Sigma'(\xi, \tau) = \exp\left[\frac{\tau}{\lambda(1+\lambda)}\right]\Bigg(\operatorname{erfc}\left[\frac{\xi(1+\lambda)^{1/2}}{2(\lambda\tau)^{1/2}}\right]$$

$$-\frac{1}{\lambda(1+\lambda)}\int_0^\tau \left\{J_0\left[\frac{2(\eta)^{1/2}(\tau-\eta)^{1/2}}{\lambda(1+\lambda)^{1/2}}\right] + \frac{(1+\lambda)^{1/2}(\eta)^{1/2}}{(\tau-\eta)^{1/2}} J_1\left[\frac{2(\eta)^{1/2}(\tau-\eta)^{1/2}}{\lambda(1+\lambda)^{1/2}}\right]\right\}$$

$$\cdot \exp\left[\frac{-(2+\lambda)(\tau-\eta)}{\lambda(1+\lambda)}\right]\operatorname{erfc}\left[\frac{\xi(1+\lambda)^{1/2}}{2(\lambda\eta)^{1/2}}\right]d\eta\Bigg). \tag{B.14}$$

Appendix C. Equation (21) for the velocity becomes, in dimensionless form,

$$V_{\tau\tau} + kV_{\tau\tau\tau} = V_{\xi\xi} + (1+k)V_{\xi\xi\tau}\,. \tag{C.1}$$

Equations (A.2) and (A.3) again hold and applying the Laplace transform we obtain

$$q^2(1+kq)L\{V\} = [1+(1+k)q]L\{V_{\xi\xi}\}, \tag{C.2}$$

together with Eqs. (A.6) and (A.7). Solving Eq. (C.2) for $L\{V(\xi,\tau)\}$ and applying the boundary conditions of (A.7),

$$L\{V(\xi,\tau)\} = \frac{1}{q}\exp\left\{\frac{-\xi q(1+kq)^{1/2}}{[1+(1+k)q]^{1/2}}\right\}. \tag{C.3}$$

From Eq. (A.6) and the boundary condition in (A.7),

$$L\{\Sigma(\xi,\tau)\} = \frac{[1+(1+k)q]^{1/2}}{q(1+kq)^{1/2}}\exp\left\{\frac{-\xi q(1+kq)^{1/2}}{[1+(1+k)q]^{1/2}}\right\}. \tag{C.4}$$

Let

$$\exp\left\{\frac{-\xi q(1+kq)^{1/2}}{[1+(1+k)q]^{1/2}}\right\} = L\{A(\xi,\tau)\}, \tag{C.5}$$

so that $A(\xi, \tau)$ is the dimensionless acceleration. Then,

$$V(\xi,\tau) = \int_0^\tau A(\xi,\zeta)\,d\zeta, \tag{C.6}$$

and

$$\Sigma(\xi,\tau) = \int_\xi^\infty A(\zeta,\tau)\,d\zeta. \tag{C.7}$$

Let

$$\alpha = \frac{(2k+1)}{2k(k+1)}, \qquad \beta = \frac{1}{2k(k+1)}, \qquad \omega = \frac{\xi(k)^{1/2}}{(1+k)^{1/2}}\,\cdot \tag{C.8}$$

Then,

$$\exp\left[-\omega q\frac{(q+\alpha+\beta)^{1/2}}{(q+\alpha-\beta)^{1/2}}\right] = \exp\left\{-\frac{\xi q(1+kq)^{1/2}}{[1+(1+k)q]^{1/2}}\right\} \tag{C.9}$$

Now,

$$\exp\left[-\omega(q-\alpha)\frac{(q+\beta)^{1/2}}{(q-\beta)^{1/2}}\right] = \exp(-\omega q)\exp[-\omega(\alpha-2\beta)]\exp\left[-\omega\left\{[(q^2-\beta^2)^{1/2} - q+\beta] - \frac{2\beta(\alpha-\beta)}{[(q^2-\beta^2)^{1/2}-q+\beta]}\right\}\right]. \tag{C.10}$$

Also,

$$\{\exp[-\eta(q^2-\beta^2)^{1/2}] - \exp(-\eta q)\} = L\left\{\frac{\beta\eta}{(\tau^2-\eta^2)^{1/2}} I_1[\beta(\tau^2-\eta^2)^{1/2}]H(\tau-\eta)\right\}, \tag{C.11}$$

so that

$$\exp[-\eta\{(q^2-\beta^2)^{1/2} - q+\beta\}] = L\left\{e^{-\beta\eta}\left[\delta(\tau) + \frac{\beta\eta}{(\tau)^{1/2}(\tau+2\eta)^{1/2}} I_1\{\beta(\tau)^{1/2}(\tau+2\eta)^{1/2}\}H(\tau)\right]\right\}. \tag{C.12}$$

From (C.12) and (A.10) it follows that, if $F(q) = L\{f(\tau)\}$,

$$F[(q^2-\beta^2)^{1/2}-q+\beta] = L\left\{F(\beta)\delta(\tau) + \beta\int_0^\infty e^{-\beta\eta}\eta\,\frac{I_1[\beta(\tau)^{1/2}(\tau+2\eta)^{1/2}]}{(\tau)^{1/2}(\tau+2\eta)^{1/2}} f(\eta)\,d\eta\right\}. \tag{C.13}$$

But,

$$\exp\left[-\omega\left\{q - \frac{1}{k(k+1)^2 q}\right\}\right] = L\left\{\delta(\tau-\omega) + \frac{(\omega)^{1/2}}{(k)^{1/2}(k+1)(\tau-\omega)^{1/2}} I_1\left[\frac{2(\omega)^{1/2}(\tau-\omega)^{1/2}}{(k)^{1/2}(k+1)}\right]H(\tau-\omega)\right\}. \tag{C.14}$$

Thus, since

$$2\beta(\alpha-\beta) = \frac{1}{k(k+1)^2},$$

$$\exp\left[-\omega\left\{[(q^2-\beta^2)^{1/2}-q+\beta] - \frac{2\beta(\alpha-\beta)}{[(q^2-\beta^2)^{1/2}-q+\beta]}\right\}\right]$$
$$= L\left\{\exp[\omega(2\alpha-3\beta)]\delta(\tau) + \beta\omega e^{-\beta\omega}\frac{I_1[\beta(\tau)^{1/2}(\tau+2\omega)^{1/2}]}{(\tau)^{1/2}(\tau+2\omega)^{1/2}} + \beta\int_\omega^\infty e^{-\beta\eta}\,\eta\,\frac{I_1[\beta(\tau)^{1/2}(\tau+2\eta)^{1/2}]}{(\tau)^{1/2}(\tau+2\eta)^{1/2}} \cdot \frac{(\omega)^{1/2}}{(k)^{1/2}(1+k)} I_1\left[\frac{2(\omega)^{1/2}(\eta-\omega)^{1/2}}{(k)^{1/2}(1+k)}\right]\frac{d\eta}{(\eta-\omega)^{1/2}}\right\}. \tag{C.15}$$

Hence, from (C.10) and (C.15),

$$\exp\left[-\omega(q-\alpha)\frac{(q+\beta)^{1/2}}{(q-\beta)^{1/2}}\right] = L\Big\{\exp[\omega(\alpha-\beta)]\delta(\tau-\omega)$$

$$+\beta\omega\exp[-\omega(\alpha-\beta)]\frac{I_1[\beta(\tau^2-\omega^2)^{1/2}]}{(\tau^2-\omega^2)^{1/2}}H(\tau-\omega)$$

$$+\beta\exp[-\omega(\alpha-\beta)]H(\tau-\omega)\int_0^\infty e^{-\beta\eta}(\eta+\omega)\frac{I_1[\beta\{(\tau+\eta)^2-(\eta+\omega)^2\}^{1/2}]}{[(\tau-\eta)^2-(\eta+\omega)^2]^{1/2}}$$

$$\cdot\frac{(\omega)^{1/2}}{(k)^{1/2}(1+k)}I_1\left[\frac{2(\omega\eta)^{1/2}}{(k)^{1/2}(1+k)}\right]\frac{d\eta}{(\eta)^{1/2}}\Big\}. \tag{C.16}$$

Thus finally, using the shift theorem, from (C.5), (C.8) and (C.9)

$$A(\xi,\tau) = \exp\left[\frac{-(2k+1)\tau}{2k(k+1)}\right]\Big\{\exp\left[\frac{(k)^{1/2}\xi}{(k+1)^{3/2}}\right]\delta\left[\tau-\frac{(k)^{1/2}\xi}{(k+1)^{1/2}}\right]$$

$$+\frac{1}{2(k)^{1/2}(k+1)^{3/2}}\exp\left[-\frac{(k)^{1/2}\xi}{(k+1)^{3/2}}\right]H\left[\tau-\frac{(k)^{1/2}\xi}{(k+1)^{1/2}}\right]$$

$$\cdot\left[\frac{\xi}{\left\{\tau^2-\dfrac{k\xi^2}{(k+1)}\right\}^{1/2}}I_1\left[\frac{\left\{\tau^2-\dfrac{k\xi^2}{(k+1)}\right\}^{1/2}}{2k(k+1)}\right]\right.$$

$$+\int_0^\infty\exp\left[\frac{-\eta}{2(k)^{1/2}(k+1)^{3/2}}\right]\frac{(\eta+\xi)(\xi)^{1/2}}{(k+1)^{3/2}(\eta)^{1/2}}I_1\left[\frac{2(\xi\eta)^{1/2}}{(k+1)^{3/2}}\right]$$

$$\cdot I_1\left[\frac{\left\{\left[\tau+\dfrac{(k)^{1/2}\eta}{(k+1)^{1/2}}\right]^2-\dfrac{k(\xi+\eta)^2}{(k+1)}\right\}^{1/2}}{2k(k+1)}\right]$$

$$\left.\cdot\frac{d\eta}{\left\{\left[\tau+\dfrac{(k)^{1/2}\eta}{(k+1)^{1/2}}\right]^2-\dfrac{k(\xi+\eta)^2}{(k+1)}\right\}^{1/2}}\right]. \tag{C.17}$$

If we let

$$B(\xi,\tau) = A(\xi,\tau)-\exp\left[\frac{-(2k+1)\tau}{2k(k+1)}+\frac{(k)^{1/2}\xi}{(k+1)^{3/2}}\right]\delta\left[\tau-\frac{(k)^{1/2}\xi}{(k+1)^{1/2}}\right], \tag{C.18}$$

then from (C.6), (C.7) and (C.17),

$$\Sigma'(\xi,\tau) = V(\xi,\tau) = \Big\{\exp\left[\frac{-\xi}{2(k)^{1/2}(k+1)^{3/2}}\right]$$

$$+\int_{[(k)^{1/2}\xi]/[(k+1)^{1/2}]}^{\tau}B(\xi,\zeta)\,d\zeta\Big\}H\left[\tau-\frac{(k)^{1/2}\xi}{(k+1)^{1/2}}\right], \tag{C.19}$$

and

$$\Sigma(\xi,\tau) = \Big\{\frac{(k+1)^{1/2}}{(k)^{1/2}}\exp\left[\frac{-\tau}{2k(k+1)}\right]$$

$$+\int_\xi^{[(k+1)^{1/2}\tau]/(k)^{1/2}}B(\zeta,\tau)\,d\zeta\Big\}H\left[\tau-\frac{(k)^{1/2}\xi}{(k+1)^{1/2}}\right]. \tag{C.20}$$

From (C.17) and (C.18) we obtain

$$\lim_{[\tau\to[(k)^{1/2}\xi]/[(k+1)^{1/2}]+]}\{B(\xi,\tau)\} = \frac{(4k+1)\xi}{8(k)^{3/2}(k+1)^{5/2}}\exp\left[\frac{-\xi}{2(k)^{1/2}(k+1)^{3/2}}\right]$$
$$= \lim_{[\tau\to[(k)^{1/2}\xi]/[(k+1)^{1/2}]+]}\left\{\frac{\partial\Sigma'}{\partial\tau}(\xi,\tau)\right\}, \qquad \text{(C.21)}$$

and

$$\lim_{[\xi\to[(k+1)^{1/2}\tau]/(k)^{1/2}-]}\{B(\xi,\tau)\} = \frac{(4k+1)\tau}{8(k)^2(k+1)^2}\exp\left[\frac{-\tau}{2k(k+1)}\right]$$
$$= \lim_{[\xi\to[(k+1)^{1/2}\tau]/(k)^{1/2}-]}\left\{-\frac{\partial\Sigma}{\partial\xi}(\xi,\tau)\right\}. \qquad \text{(C.22)}$$

Also, from (C.19) and (C.20),

$$\lim_{[\tau\to[(k)^{1/2}\xi]/[(k+1)^{1/2}]+]}\left\{\frac{\partial\Sigma}{\partial\tau}(\xi,\tau)\right\}$$
$$= \frac{[(4k+1)\xi - 4(k)^{1/2}(k+1)^{3/2}]}{8(k)^2(k+1)^2}\exp\left[\frac{-\xi}{2(k)^{1/2}(k+1)^{3/2}}\right], \qquad \text{(C.23)}$$

and

$$\lim_{[\xi\to[(k+1)^{1/2}\tau]/(k)^{1/2}-]}\left\{\frac{\partial\Sigma'}{\partial\xi}(\xi,\tau)\right\} = \frac{[(4k+1)\tau + 4k(k+1)]}{8(k)^{3/2}(k+1)^{5/2}}\exp\left[\frac{-\tau}{2k(k+1)}\right].$$

Further,

$$\left.\begin{aligned}\lim_{[\xi\to[(k+1)^{1/2}\tau]/(k)^{1/2}-]}\{\Sigma'(\xi,\tau)\} &= \exp\left[\frac{-\tau}{2k(k+1)}\right],\\ \lim_{[\xi\to[(k+1)^{1/2}\tau]/(k)^{1/2}-]}\{\Sigma(\xi,\tau)\} &= \frac{(k+1)^{1/2}}{(k)^{1/2}}\exp\left[\frac{-\tau}{2k(k+1)}\right].\end{aligned}\right\} \qquad \text{(C.24)}$$

From Eq. (C.4),

$$L\{\Sigma(0,\tau)\} = \frac{[1+(1+k)q]^{1/2}}{q(1+kq)^{1/2}}$$
$$= \frac{\left[(1+k)+\dfrac{1}{q}\right]}{(k)^{1/2}(1+k)^{1/2}}\frac{1}{\left\{\left[q+\dfrac{(2k+1)}{2k(k+1)}\right]^2 - \dfrac{1}{4k^2(k+1)^2}\right\}^{1/2}},$$

so that

$$\Sigma(0,\tau) = \frac{(1+k)^{1/2}}{(k)^{1/2}}\left\{\exp\left[\frac{-(2k+1)\tau}{2k(k+1)}\right]I_0\left[\frac{\tau}{2k(k+1)}\right]\right.$$
$$\left. + \frac{1}{(1+k)}\int_0^\tau \exp\left[\frac{-(2k+1)\eta}{2k(k+1)}\right]I_0\left[\frac{\eta}{2k(k+1)}\right]d\eta\right\}. \qquad \text{(C.25)}$$

In particular, $\Sigma(0,\infty) = 1$.

BOOK REVIEWS

Theoretical elasticity. By A. E. Green and W. Zerna. Oxford University Press, New York, 1954. xiii + 442 pp. $8.00.

All serious students of the mathematical theory of elasticity will welcome this book. It does not make any claim to being a treatise on the subject. Rather, it represents both in choice of subject and presentation the personal interests and point-of-view of the authors. Indeed many of the topics discussed are ones which are to a great extent the results of Professor Green's own researches. Much of the material has not been previously presented in a unified notation or in book form.

The subjects covered are drawn both from the finite and infinitesimal elasticity theories. In presenting the basic concepts of these theories, finite elasticity theory is developed first, and the fundamental formulation of the infinitesimal theory is derived as a first approximation to the more general theory. Although this no doubt adds to the formal complexity of the presentation, it has the undoubted advantages of rigor and conceptual clarity.

The finite elasticity theory is used to solve a number of simple problems which admit of exact general solution without further approximation or assumption—except that of incompressibility of the material. The theory of the superposition of infinitesimal deformations on finite deformations is then developed.

The infinitesimal theory is developed for the cases of plane strain and generalised plane stress and is applied to the discussion of a wide range of problems for both isotropic and anisotropic materials. Complex variable methods are used consistently in a manner somewhat similar to that adopted by Muskhelishvili. In the final four chapters of the book the general bending theory of shells and membrane theory are briefly discussed, together with their particularisation to cylindrical shells and shells of revolution.

Throughout the book tensor notation with convected coordinates is used. Although this makes for great elegance of presentation, it will no doubt limit to a large degree the extent to which the book will be used.

R. S. Rivlin

Stochastic models for learning. By Robert R. Bush and Frederick Mosteller. John Wiley and Sons, Inc., New York, and Chapman & Hall, Ltd., London, 1955. xvi + 365 pp. $9.00.

This book is primarily concerned with models for analysing learning experiments in which the learner is exposed repeatedly to the same stimulus. The essential ingredients of the model are relations between stimuli, responses and events. It is postulated that there is a set of probabilities associated with the various responses to the given stimulus. Following each response a certain event takes place such as for example a rewarding or an absence of a reward for the subject and this changes the response probabilities for future stimuli. The event will generally depend in some preassigned way upon the response, for example there may be a one to one correspondence between events and the responses. The events are represented by linear transformations to be applied to the response probabilities.

Some very interesting mathematical problems arise in connection with the determination of the asymptotic behavior of the response probabilities after a large number of repetitions of the stimulus. Although the events themselves are represented by linear transformations on the probabilities, which event takes place depends upon the response to the previous stimulus; this in turn is a function of the response probabilities. The probabilities after any event are generally quadratic functions of those after the previous event.

Although the book contains many details that will be useful only to a psychologist who wants perhaps actually to apply the model to some data, it also contains numerous unsolved problems that make it very interesting reading for the non-specialist and it should greatly stimulate future study by both mathematicians and psychologists.

G. F. Newell

(Continued on p. 220)

SPHERICAL, CYLINDRICAL AND ONE-DIMENSIONAL GAS FLOWS*

BY

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Abstract. The problem of spherical, cylindrical or planar flow of a polutropic gas has been formulated in Lagrangian variables, giving rise to a non-linear second order partial differential equation in two variables, h and t. A class of special solutions of this equation has been found by separation of variables, and these solutions depend upon an arbitrary function which is related to the arbitrary entropy distribution in the gas. By specializing this function solutions corresponding to the expansion into a vacuum of an isentropic or non-isentropic gas cloud have been obtained as well as solutions corresponding to the propagation of finite and of strong shocks in variable media.

1. Introduction. Relatively few boundary value problems involving spherical or cylindrical flows of gases or compressible fluids have been solved exactly. In order to solve more problems of this type, particularly those involving variable entropy, we have investigated the flow differential equations, at first without regard to initial or boundary conditions. In this way a class of solutions of the differential equations, depending upon an arbitrary function, has been obtained. In general the solutions of this class represent flows with variable entropy, although some isentropic solutions are included. Then, the arbitrary function is adjusted to satisfy particular initial or boundary conditions. In this way the free expansion into a vacuum of a sphere, cylinder or slab of gas has been treated, as well as the propagation of finite and strong shocks in variable media. The latter treatment includes Primakoff's point-blast solution as a special case. The expansion of a sphere of gas into a vacuum may be of interest to astrophysicists who have treated corresponding one-dimensional problems.

The problem to be solved is formulated in Lagrangian variables, in Sec. 2 of this paper. In Sec. 3 a class of solutions is obtained by the method of separation of variables and in Sec. 4 the isentropic solutions are examined. In Sec. 5 the particle paths of the flows represented by the solutions are analyzed. In Sec. 6 the solutions are used to describe the expansion of a gas into a vacuum, in Sec. 7 to describe strong shock waves propagating in variable media and in Sec. 8 to describe finite shocks in variable media. Section 9 contains conclusions and extensions of the results.

2. Formulation. We consider the motion of an inviscid, non-heat-conducting fluid, obeying the polytropic equation of state. The one, two, and three-dimensional cases will be treated together. In the three-dimensional case $y(h, t)$ represents the radius at time t of the particle with the Lagrangian coordinate h, which is defined by the equation[1]

$$h = \int_{y(0,t)}^{y(h,t)} r^{n-1}\rho(r, t)\, dr \qquad n = 1, 2, 3. \tag{1}$$

In the above equation $\rho(r, t)$ is the density at time t and radius r, and n is the dimension, which is 3 in the spherical case. In the two dimensional case y represents radial distance from an axis, and in one dimension y is a cartesian coordinate.

*Received June 22, 1955. This work was sponsored by the Office of Naval Research under Contract No. Nonr-285(02).

[1]R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Interscience Press, 1948, pp. 30–32.

It is further assumed that all flow variables depend upon y and t only and that flow occurs only in the y direction. This is the assumption of spherical, cylindrical or planar symmetry.

From the definition of y, the particle velocity u is given by

$$u = y_t \,. \tag{2}$$

Similarly from (1) the density ρ and specific volume τ are given by

$$\tau = \rho^{-1} = y^{n-1}y_h \,. \tag{3}$$

Because of our assumptions concerning inviscidity and non-conduction, the entropy s of a particle is independent of time (at least between successive shocks). Thus we have

$$s = s(h). \tag{4}$$

The function $s(h)$ is given by initial data or by shock conditions, and is assumed to be known.

The pressure p is given by the equation of state

$$p = p(\rho, s) = g(\tau, s). \tag{5}$$

For a polytropic gas or liquid

$$g(\tau, s) = g_0 + A(s)\tau^{-\gamma}. \tag{6}$$

The function $A(s)$, the adiabatic exponent γ, and the internal pressure g_0 are assumed to be known.

In terms of the above defined quantities, the equation of motion is

$$y_{tt} = -y^{n-1}[g_\tau(y^{n-1}y_h)_h + g_s s_h]. \tag{7}$$

For a polytropic gas or liquid this becomes, using (6),

$$y_{tt} = \gamma A(s)(y^{n-1}y_h)^{-\gamma-1}(y^{n-1}y_h)_h y^{n-1} - A_h(y^{n-1}y_h)^{-\gamma}y^{n-1}. \tag{8}$$

Equation (8) is a second order partial differential equation for $y(h, t)$. The coefficient A is assumed to be a known function of s, and by (4), of h.

The problem we consider is that of finding solutions of (8). From a particular solution, the flow variables can be found by using (2), (3), (4) and (5).

3. Product solutions. Let us seek product solutions of (8) of the form

$$y(h, t) = f(h)j(t). \tag{9}$$

Inserting (9) into (8), and separating variables we obtain

$$j'' - \lambda j^{n(1-\gamma)-1} = 0 \tag{10}$$

$$-A[(f^{n-1}f')^{-\gamma}]'f^{n-2} - A'(f^{n-1}f')^{-\gamma}f^{n-2} = \lambda. \tag{11}$$

In these equations λ is an arbitrary separation parameter and primes represent derivatives with respect to t in (10) and with respect to h in (11).

To solve (10) we multiply by j' and integrate, obtaining

$$(j')^2 = \frac{2\lambda}{n(1-\gamma)} j^{n(1-\gamma)} + a \qquad (\gamma \neq 1) \tag{12}$$

$$(j')^2 = 2\lambda \log j + a \qquad (\gamma = 1). \tag{13}$$

Here a is an integration constant. Thus, unless j is constant, which is possible only if $\lambda = a = 0$ or if $j = 0$, we have the solution

$$\int_{j_0}^{j} \left[\frac{2\lambda}{n(1-\gamma)} j^{n(1-\gamma)} + a\right]^{-1/2} dj = t \qquad (\gamma \neq 1) \tag{14}$$

$$\int_{j_0}^{j} [2\lambda \log j + a]^{-1/2}\, dj = t \qquad (\gamma = 1). \tag{15}$$

To solve (11) we differentiate out and obtain

$$\gamma A[f''f^{n-1} + (n-1)f^{n-2}(f')^2](f^{n-1}f')^{-\gamma-1}f^{n-2} - f^{n-2}(f^{n-1}f')^{-\gamma}A' = \lambda. \tag{16}$$

We now consider the inverse function $h = h(f)$ and denote $h'(f)$ by $q(f)$. Then (16) becomes

$$\gamma A[-q'q^{-3}f^{n-1} + (n-1)f^{n-2}q^{-2}][f^{n-1}q^{-1}]^{-\gamma-1}f^{n-2} - f^{n-2}[f^{n-1}q^{-1}]^{-\gamma}A'(h) = \lambda. \tag{17}$$

We shall first treat the case $\gamma \neq 1$ by introducing $z(f)$ and $B(f)$ defined by

$$z = q^{\gamma-1}, \qquad B(f) = A[h(f)]. \tag{18}$$

With these definitions, (17) becomes (prime denoting differentiation with respect to f)

$$z' + z\left[-(n-1)(\gamma-1)f^{-1} + \frac{\gamma-1}{\gamma}(\log B)'\right] + \frac{\lambda(\gamma-1)}{\gamma B} f^{(n-1)(\gamma-1)+1} = 0. \tag{19}$$

The solution of (19) is (with G a constant)

$$z = f^{(n-1)(\gamma-1)}B^{(1-\gamma)/\gamma}\left[G - \frac{\lambda(\gamma-1)}{\gamma}\int^{f} fB^{-1/\gamma}\, df\right]. \tag{20}$$

Thus we have q from (18) and (20), and finally since $h' = q$,

$$h = \int_{f_0}^{f} f^{n-1}B^{-1/\gamma}\left[G - \frac{\lambda(\gamma-1)}{\gamma}\int^{f} fB^{-1/\gamma}\, df\right]^{\gamma/(\gamma-1)} df. \tag{21}$$

Equation (21) gives $f(h)$ implicitly, and thus provides a solution of (11) for $\gamma \neq 1$. The solution may be written more simply by defining $F(f)$ by

$$F(f) = \left[G - \frac{\lambda(\gamma-1)}{\gamma}\int^{f} fB^{-1/\gamma}\, df\right]^{\gamma/(\gamma-1)}. \tag{22}$$

If $\lambda \neq 0$ this can be solved for $B(f)$ and yields

$$B(f) = (-\lambda f)^{\gamma}(f')^{-\gamma}F. \tag{23}$$

The solution for the flow variables can now be computed from (2)-(5), (9), (21), (22). We have for $\gamma \neq 1$ and $\lambda \neq 0$, remembering that $f = yj^{-1}$ from (9),

$$u(y, t) = yj'j^{-1} \tag{24}$$

$$\tau(y, t) = -\lambda yj^{n-1}/F'(yj^{-1}) \tag{25}$$

$$p(y, t) = g_0 + j^{-n\gamma}F(yj^{-1}). \tag{26}$$

Equations (24)-(26) give the flow quantities. In fact, these expressions yield a solution of the Eulerian equations of motion for an arbitrary function F provided that $j(t)$ is given by (14). This is the first main result of this paper. It is to be noted that in these solutions, u is proportional to y and that F must be so chosen that τ, given by (25), is positive. This implies that F must be monotonic in a region where y is of one sign.

In the excluded case $\gamma \neq 1$, $\lambda = 0$ we have instead of (24)-(26) from (2)-(5), (9), and (21)

$$u(y, t) = yj'j^{-1} \qquad = yt^{-1}, \tag{27}$$

$$\tau(y, t) = j^n B^{1/\gamma}(yj^{-1})G^{1/(1-\gamma)} = t^n b(yt^{-1}), \tag{28}$$

$$p(y, t) = g_0 + j^{-n\gamma}G^{\gamma/(1-\gamma)} \qquad = g_0 + lt^{-n\gamma}. \tag{29}$$

The new arbitrary function b and arbitrary constant l have been introduced and the expressions simplified by using (14), which gives

$$j(t) = \pm a^{1/2}t. \tag{30}$$

The origin of t is chosen so that $j(0) = 0$.

The corresponding solutions when $\gamma = 1$ are obtained from (17) by introducing $B(f)$ as in (18), after which (17) becomes

$$q'q^{-1} - (n-1)f^{-1} + \frac{B'}{B} + \frac{\lambda f}{B} = 0. \tag{31}$$

The solution of (31) is

$$q(f) = f^{n-1}B^{-1}(f) \exp \int^f - \lambda f B^{-1}(f)\, df. \tag{32}$$

Thus

$$h(f) = \int_{f_0}^{f} \left[f^{n-1}B^{-1}(f) \exp \int^f - \lambda f B^{-1}(f)\, df \right] df. \tag{33}$$

Equation (33) yields the solution implicitly. Again we define

$$F(f) = \exp \int^f - \lambda f B^{-1}(f)\, df. \tag{34}$$

Then if $\lambda \neq 0$ we have

$$B(f) = -\lambda f F(F')^{-1}. \tag{35}$$

The solution for the flow variables, if $\gamma = 1$ and $\lambda \neq 0$, is again given by (24)-(26) with $\gamma = 1$ and $j(t)$ given by (15). Similarly for $\gamma = 1$ and $\lambda = 0$ the solution is given by (27)-(29) with $\gamma = 1$, $G = 1$ and $j(t)$ given by (30).

Equations (24)-(26) and (27)-(29) represent non-isentropic solutions of the flow equations depending upon an arbitrary function. In order to construct these solutions explicitly one need merely evaluate the integrals in (14) or (15).

4. Isentropic case. The above solution can be specialized to the isentropic case by setting $A = B =$ a constant. Then from (22) and (34) we obtain

$$F(f) = \left[G - \frac{\lambda(\gamma - 1)}{2\gamma B^{1/\gamma}} f^2 \right]^{\gamma/(\gamma-1)} \qquad \gamma \neq 1, \tag{36}$$

$$F(f) = \exp \frac{-\gamma f^2}{2B} \qquad \gamma = 1. \tag{37}$$

Thus the solutions given by (24)-(26) become, for $\gamma \neq 1$ and $\lambda \neq 0$

$$u(y, t) = yj'j^{-1}, \tag{38}$$

$$\tau(y, t) = j^n B^{1/\gamma}\left[G - \frac{\lambda(\gamma - 1)}{2\gamma B^{1/\gamma}}\, y^2 j^{-2}\right]^{-1/(\gamma-1)}, \tag{39}$$

$$p(y, t) = g_0 + j^{-n\gamma}\left[G - \frac{\lambda(\gamma - 1)}{2\gamma B^{1/\gamma}}\, y^2 j^{-2}\right]^{\gamma/(\gamma-1)}. \tag{40}$$

In these solutions B and G are constants and j is given by (14). For $\lambda = 0$, and all γ the solution (27)-(29) applies with b constant.

For $\gamma = 1$ and all λ, (38) is unchanged, but the other equations become

$$\tau(y, t) = j^n B \exp \frac{-\lambda j^{-2} y^2}{2B}, \tag{41}$$

$$p(y, t) = g_0 + j^{-n} \exp \frac{\lambda j^{-2} y^2}{2B}. \tag{42}$$

Here B is a constant and j is given by (15).

As an example of these isentropic solutions, let $j(t)$ be given by (48) and assume $G = 0$ in (39) and (40). Then (38)-(40) yield the power solutions (if $\gamma \neq 1$, $n(\gamma - 1)/2 \neq -1$

$$u(y, t) = \frac{2}{n(\gamma - 1) + 2}\, yt^{-1}, \tag{43}$$

$$\tau(y, t) = \left[\frac{\gamma B}{n}\left(\frac{n(\gamma - 1) + 2}{\gamma - 1}\right)^2\right]^{1/(\gamma-1)} (yt^{-1})^{-2/(\gamma-1)}, \tag{44}$$

$$p(y, t) = g_0 + B\left[\frac{\gamma B}{n}\left(\frac{n(\gamma - 1) + 2}{\gamma - 1}\right)^2\right]^{\gamma/(1-\gamma)} (yt^{-1})^{2\gamma/(\gamma-1)}. \tag{45}$$

5. Particle paths. A particle designated by a fixed value of h, follows the path given by (9), i.e. $y = f(h)j(t)$ where $j(t)$ is a solution of (12) if $\gamma \neq 1$ and of (13) if $\gamma = 1$. Let us first consider the case in which the constant λ in (12) and (13) is zero. Then in both cases we have

$$j(t) = j(0) \pm a^{1/2} t. \tag{46}$$

If the origin of t is shifted so that $j(0) = 0$ then this becomes

$$j(t) = \pm a^{1/2} t. \tag{47}$$

This solution has already been used to simplify (27)-(29), [see Eq. (30)]. In the motion described by (44) with the plus sign, all the particles start from the origin at $t = 0$ and each moves outward with the constant speed $a^{1/2} f(h)$. With the minus sign all particles move toward the origin, each with the constant speed $-a^{1/2} f(h)$ and they all arrive there at $t = 0$.

Now let us examine the case $\gamma > 1$, $\lambda > 0$. Then in order that $(j')^2$ in (12) be positive, we must have the constant a in (12) positive. From (12) we see that the solution is then

of the form shown in Fig. 1 when the origin of t is chosen so that $j'(0) = 0$. This choice of origin requires that the constant $a = -2\lambda j_0^{n(1-\gamma)}/[n(1-\gamma)]$.

This solution is even in t. Each particle has the constant speed $-a^{1/2}f(h)$ at $t = -\infty$ and slows down until it comes to momentary rest at $t = 0$ with $y = f(h)j(0)$, after which

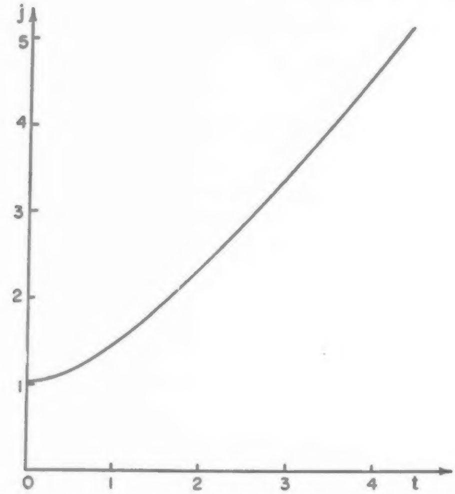


FIG. 1. The behavior of $j(t)$ for $\gamma > 1$, $\lambda > 0$. The origin of t is so chosen that $j'(0) = 0$ which requires $a = -2\lambda j(0)^{n(1-\gamma)}/[n(\gamma - 1)]$. At $t \to \pm\infty$, $j' \to \mp a^{1/2}$. The curve is obtained from Eq. (14) with $\gamma = 1.4$, $n = 3$, $\lambda = j(0) = 1$.

it again moves outward ultimately approaching the speed $+a^{1/2}f(h)$. The previous solution in which $j(0) = 0$ is obviously a limiting form of this solution. If we only consider that half the solution $t \geq 0$ we have the expansion of a gas initially at rest.

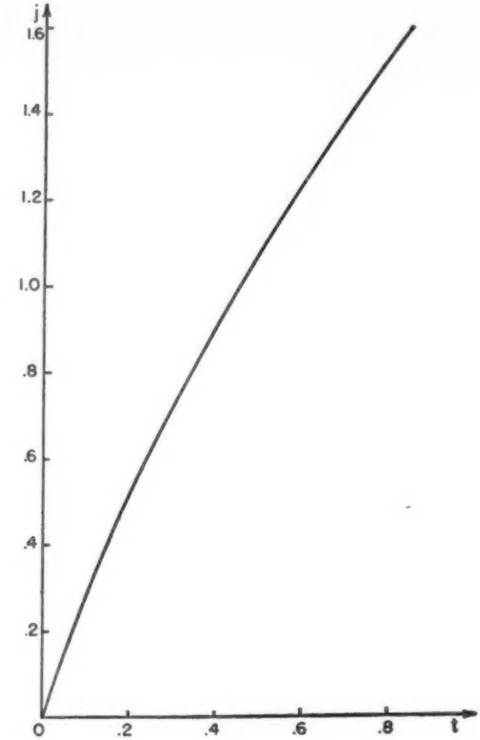
The other possibility when $\gamma > 1$ is that $\lambda < 0$. The nature of the solution now depends upon the sign of a as is shown in Fig. 2 which is obtained by analyzing (12). In the first case ($a > 0$) each particle has the constant speed $-a^{1/2}f(h)$ at $t = -\infty$ and speeds up until it reaches the origin at $t = 0$ with infinite speed. Thus all the gas collapses into the origin. The positive branch of the curve describes the reverse phenomenon —all the gas is released from the origin at $t = 0$ with infinite speed and it spreads out and slows down, each particle approaching the constant speed $a^{1/2}f(h)$ as t becomes infinite. The second case ($a = 0$) is similar to the first, but the speed approached as t approaches plus or minus infinity is zero. The explicit solution in this case is

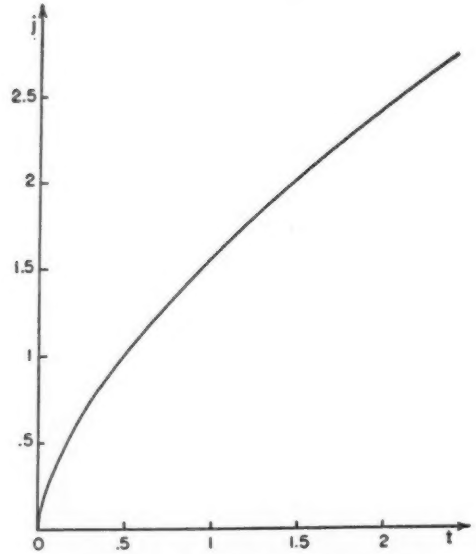
$$j(t) = \left\{\left[\frac{2\lambda}{n(1-\gamma)}\right]^{1/2}\left(\frac{n(\gamma-1)+2}{2}\right)t\right\}^{2/[n(\gamma-1)+2]}$$

$$\text{if } \gamma \neq 1,\ a = 0,\ \tfrac{1}{2}n(\gamma - 1) \neq -1. \qquad (48)$$

The third case ($a < 0$) is different from the preceding cases. The solution is periodic, all the gas being released from the origin at some instant, expanding, coming to rest, collapsing into the origin again, etc.

When $\gamma = 1$ and $\lambda < 0$ the solution has, for all values of a, the behavior shown in Fig. 2c while for $\lambda > 0$ it has the behavior shown in Fig. 1, as can be seen from (13).





Figs. 2a and b. (For legend see p. 178.)

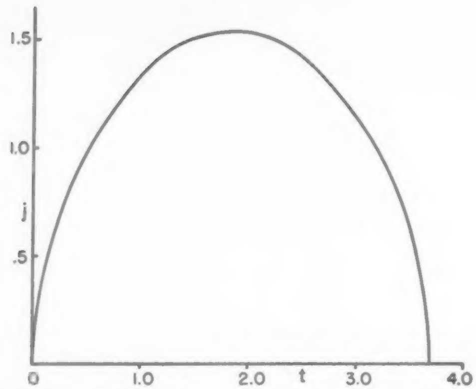


FIG. 2c. The behavior of $j(t)$ for $\gamma > 1$ and $\lambda > 0$. The origin of t is so chosen that $j(0) = 0$. In the first case, $a > 0$, j' approaches $\mp a^{1/2}$ as $t \to \pm \infty$. In the second case $a = 0$, j' approaches zero as $t \to \pm \infty$, and in the third case $a < 0$, j is periodic in t. The curves are computed for $\gamma = 1.4$, $n = 3$, $\lambda = -1$, $a = 1$ in the first case and $a = -1$ in the third case, using Eq. (14). In the second case $a = 0$, the solution is given by (48).

6. Application: Free expansion of gas into vacuum. Let us apply the isentropic solution (38-40) to the free expansion into a vacuum of a gas with $\gamma > 1$. Then the pressure must be zero for some value of h, which represents the interface between gas and vacuum. If the interface is initially at $y = y_0$, and if $g_0 = 0$ since the medium is a gas, we need merely set $G = [\lambda(\gamma - 1)]y_0^2 j(0)^{-2}/2\gamma B^{1/\gamma}$ in (39, 40). Then $p = 0$ at the interface which is given by

$$y = y_0 \frac{j(t)}{j(0)} \cdot \tag{49}$$

The solution $j(t)$ which must be taken is of the type shown in Fig. 1, if y_0 and $j(0)$ are not zero. Thus the gas is initially at rest and, according to (40), the initial pressure decreases from a maximum at the center $y = 0$ to zero at $y = y_0$ as shown in Fig. 3. Then the gas accelerates outward, each particle approaching a constant speed while the speed at any instant is proportional to y. The pressure at any particle varies as the $-n\gamma$ power of its radius, thus decreasing as the radius increases. The preceding solution applies to the expansion into a vacuum of a sphere ($n = 3$), a cylinder ($n = 2$) or a slab ($n = 1$) of gas initially at rest, and with uniform entropy.

In a similar manner, but by using solutions for $j(t)$ of the types shown in Figs. 2a and 2b, solutions describing the expansion of an isentropic gas which fills all space except an expanding region of vacuum around the origin (i.e. a hole), can be constructed.

For a gas with $\gamma = 1$ which is also isentropic, it is possible to construct the solution for an expanding gas cloud which fills all space initially, but which has a Gaussian distribution of density. To this end we choose $\lambda > 0$, in which case $j(t)$ has the form shown in Fig. 1. Then u, t, and p are given by (38, 41, 42), with $g_0 = 0$ in (42) if the medium is a gas. This solution describes a gas cloud initially at rest, which subsequently expands, the velocity of all particles ultimately approaching the same constant value. If this solution is also considered for $t < 0$ it describes a gas cloud which contracts, comes to rest and expands again.

Another interesting motion of a gas with any value of γ results if (30) is used for

$j(t)$ with the plus sign, and if $\lambda = 0$ in (39, 40). We then obtain, setting $g_0 = 0$ in (40) to describe a gas,

$$u = yt^{-1}, \tag{50}$$

$$\tau = a^{n/2}Bt^n, \tag{51}$$

$$p = a^{-n\gamma/2}B^{1-\gamma}t^{-n\gamma}. \tag{52}$$

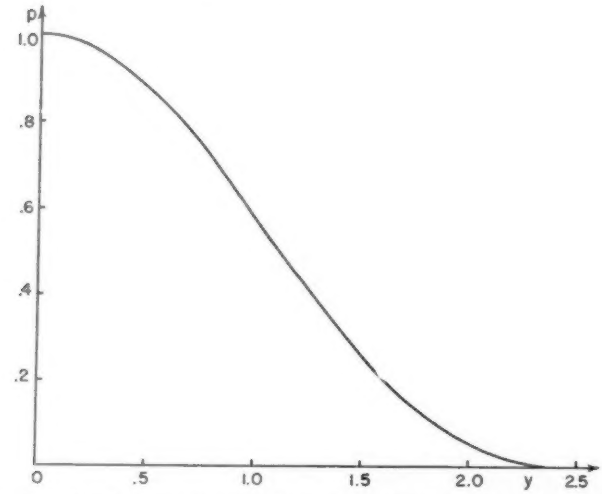


FIG. 3. Initial pressure distribution in an expanding spherical, cylindrical or planar gas cloud, computed from Eq. (40) with $g_0 = 0$, $\gamma = 1.4$, $G = B = j = 1$, $\lambda = 1$. The same curve applies at later times if the vertical scale refers to $pj^{n\gamma}$ and the horizontal scale to yj^{-1}.

This solution represents the expansion (or contraction if t is replaced by $-t$) of a gas of uniform density, pressure and entropy which fills space.

Similar motions of gases in which the entropy is not initially constant can also be constructed by using equations (24-26) to describe the flow. In order that p equal zero at the interface between gas and vacuum, F must be chosen to vanish at some value of its argument, which value will then correspond to the interface. Aside from this condition, F may be chosen arbitrarily.

7. Application: Strong shocks in variable media. Let us suppose that a shock given by the equation $y = R(t)$ moves into a variable medium of density $\rho_0(y)$, pressure $p_0(y)$ and velocity zero. The pressure p, velocity u and density ρ just behind the shock are related to the corresponding quantities in front of it by the shock conditions. These conditions are

$$\frac{\rho}{\rho_0} = \frac{(\gamma+1)p + (\gamma-1)p_0}{(\gamma-1)p + (\gamma+1)p_0} \approx \frac{\gamma+1}{\gamma-1}, \qquad (\gamma \neq \pm 1) \tag{53}$$

$$u = \frac{2(p - p_0)}{\{2\rho_0[(\gamma+1)p + (\gamma-1)p_0]\}^{1/2}} \approx \left[\frac{2p}{(\gamma+1)\rho_0}\right]^{1/2}, \tag{54}$$

$$\dot{R} = \left[\frac{(\gamma+1)p + (\gamma-1)p_0}{2\rho_0}\right]^{1/2} \approx \left[\frac{(\gamma+1)p}{2\rho_0}\right]^{1/2}. \tag{55}$$

The second expressions on the right apply to a strong shock, for which $p \gg p_0$.

We shall now assume that the flow behind the shock is a "product" solution given by (24)-(26), and that the shock is strong. We wish to determine the functions $F(yj^{-1})$, $R(t)$, $\rho_0(y)$ and $p_0(y)$ for which such a solution is possible. To this end, we insert (24)-(26) into (53)-(55) and obtain

$$\frac{F'(Rj^{-1})}{-\lambda Rj^{n-1}\rho_0(R)} = \frac{\gamma + 1}{\gamma - 1} \tag{56}$$

$$\frac{Rj'}{j} = \left[\frac{2g_0 + 2j^{-n\gamma}F(Rj^{-1})}{(\gamma + 1)\rho_0(R)}\right]^{1/2} \tag{57}$$

$$R^{\cdot} = \left[\frac{(\gamma + 1)\{g_0 + j^{-n\gamma}F(Rj^{-1})\}}{2\rho_0(R)}\right]^{1/2}. \tag{58}$$

From (57) and (58) we have

$$\frac{R^{\cdot}}{R} = \frac{(\gamma + 1)}{2} \cdot \frac{j'}{j} . \tag{59}$$

Solving, and introducing the integration constant R_0, we have for $R(t)$ the expression

$$R(t) = R_0[j(t)]^{(\gamma+1)/2}. \tag{60}$$

Inserting (60) into (57) and making use of (12), satisfied by $j(t)$, we find

$$F(x) = \frac{\gamma + 1}{2}\,\rho_0(x^{(\gamma+1)/(\gamma-1)}R_0^{2/(1-\gamma)}) \cdot \left[\frac{2\lambda}{n(1 - \gamma)}\,(xR_0^{-1})^{-2n} + D\right] \cdot R^{(-2n\gamma)/(\gamma-1)}x^{[(2n+2)\gamma-2]/(\gamma-1)} - g_0(xR_0^{-1})^{(2n\gamma)/(\gamma-1)}. \tag{61}$$

Thus R is determined by (60), F is related to ρ_0 by (61), and (57), (58) are satisfied, although the constants R_0, D, and λ are still arbitrary. We now insert (60), (61) into (56) to determine F. We obtain, if $g_0 = 0$,

$$F(x) = F_0x^n\left[\frac{2\lambda R_0^{2n}}{n(1 - \gamma)} + Dx^{2n}\right]^{-1/2}. \tag{62}$$

From (61) and (62) we then find an expression for $\rho_0(R)$, namely

$$\rho_0(R) = \frac{2F_0R_0^{(2n\gamma)/(\gamma-1)}}{\gamma + 1}\left[\frac{2\lambda R_0^{2n}}{n(1 - \gamma)} + DR_0^{(4n)/(\gamma+1)}R^{[2n(\gamma-1)]/(\gamma+1)}\right]^{-3/2} \cdot R^{[(n-2)\gamma-3n+2]/(\gamma+1)}R^{[2(n-2)\gamma-6n+4]/[(\gamma+1)(\gamma-1)]}. \tag{63}$$

In (62) and (63) F_0 is an arbitrary constant. The corresponding results with $g_0 \neq 0$ are somewhat more complicated.

We have thus obtained a solution with a strong shock moving into a variable medium at rest, with density given by (63). The constants F_0, R_0, λ and D are arbitrary in this equation, but only two essential combinations of these constants occur. The shock curve is given by (60), and $j(t)$ by (14). The flow is given by (24)-(26) with $F(x)$ given by (62). The flow might be produced by a piston following one of the particle paths

$$Y(t) = Y_0j(t). \tag{64}$$

The solution was deduced for $\gamma \neq \pm 1$ and $\lambda \neq 0$.

As an example of these solutions, let us suppose that $\rho_0(R)$ = constant. From (63) we find that this requires

$$D = 0, \qquad \gamma = \frac{3n - 2}{n - 2} \,. \tag{65}$$

The last condition can be fulfilled only for $n = 3$, in which case $\gamma = 7$; ($n = 1$ leads to $\gamma = -1$, which was excluded in the derivation, and $n = 2$ yields no value of γ). In this case we have from (14), (62)

$$j = \left[\frac{5(-\lambda)^{1/2}}{3}\, t\right]^{1/10}, \qquad F(x) = x^3 \frac{3F_0R_0^{-n}}{(-\lambda)^{1/2}} \,. \tag{66}$$

The solution computed by using (66) in (24)-(26) (with $\lambda < 0$) is exactly the point blast solution of Taylor's type first found by H. Primakoff, and applicable to high energy explosions in water. [See[1], p. 424.]

8. Application: Finite shocks in variable media. We may also apply the above method to determine finite shocks in variable media. Then we must satisfy the exact shock conditions (53)-(55) rather than the strong shock conditions. Inserting (24)-(26) into (53)-(55) yields with $g_0 = 0$

$$\frac{F'(x)}{-\lambda R j^{n-1}\rho_0} = \frac{(\gamma + 1)j^{-n\gamma}F + (\gamma - 1)p_0}{(\gamma + 1)p_0 + (\gamma - 1)j^{-n\gamma}F}\,, \qquad x = Rj^{-1} \tag{67}$$

$$Rj'j^{-1} = \frac{j^{-n\gamma}F - p_0}{(\rho_0/2)^{1/2}[(\gamma + 1)j^{-n\gamma}F + (\gamma - 1)p_0]^{1/2}}\,, \tag{68}$$

$$R^{\cdot} = \frac{1}{(2\rho_0)^{1/2}}\,[(\gamma + 1)j^{-n\gamma}F + (\gamma - 1)p_0]^{1/2}. \tag{69}$$

Equations (67)-(69) are a set of two first order ordinary differential equations and one algebraic equation for the determination of the four functions $F(x)$, $R(t)$, $p_0(R)$ and $\rho_0(R)$. Equation (14) gives $j(t)$. This system is evidently under-determined and will have infinitely many solutions. In the strong shock case, however, p_0 did not occur and thus the system was determined.

To solve the above system we could impose some relation between p_0 and ρ_0 and eliminate both these functions by means of that relation and (68). A pair of simultaneous first order equations for $F(x)$ and $R(t)$ would result. However, these equations can be treated separately since, from (68) and (69) we have by eliminating F,

$$R^{\cdot} = \frac{\gamma + 1}{4} Rj'j^{-1} \pm \left[\frac{(\gamma + 1)^2}{16}(Rj'j^{-1})^2 + \gamma p_0\rho_0^{-1}\right]^{1/2}. \tag{70}$$

This is an equation for $R(t)$ if the ratio $p_0\rho_0^{-1}$ is given as a function of R. (Equation (70) also holds when $g_0 \neq 0$.) After solving this, (67) can be solved for F, and then p_0 and ρ_0 can be obtained.

We will now investigate the shock curve $R(t)$ given by (70), in the special case in which $j(t)$ is given by (48) with $C = 0$ and $\gamma p_0 \rho_0^{-1} = c_0^2$ is constant. The latter assumption means that the temperature ahead of the shock is constant. Making use of (48), (70) becomes

$$R^{\cdot} = \frac{\gamma + 1}{2[n(\gamma - 1) + 2]} Rt^{-1} \pm \left[\frac{(\gamma + 1)^2}{4[n(\gamma - 1) + 2]^2}(Rt^{-1})^2 + c_0^2\right]^{1/2}. \tag{71}$$

If we now introduce $U(t) = Rt^{-1}$ in (71) we obtain an equation for U in which the variables separate. One solution is the constant solution

$$U = \pm c_0\left(1 - \frac{\gamma + 1}{n(\gamma - 1) + 2}\right)^{-1/2}. \tag{72}$$

In this case, which is physically possible only when $(\gamma + 1)/[n(\gamma - 1) + 2] < 1$, the shock curve is the straight line

$$R = Ut = \pm c_0\left(1 - \frac{\gamma + 1}{n(\gamma - 1) + 2}\right)^{-1/2} t. \tag{73}$$

When U does not have the value given by (72), the equation for U yields

$$\int_{U_0}^{U(t)} \left\{\left(\frac{\gamma + 1}{2n(\gamma - 1) + 4} - 1\right)U \pm \left[\frac{(\gamma + 1)^2}{[2n(\gamma - 1) + 4]^2} U^2 + c_0^2\right]^{1/2}\right\}^{-1} dU = \log \frac{t}{t_0} \cdot \tag{74}$$

Integrating (74) yields

$$bt = |\pm a + (a - 1) \cos \theta|^{(1-a)/(2a-1)}\left(\sin \frac{\theta}{2}\right)^{-1/(a \ast a-1)}\left(\cos \frac{\theta}{2}\right)^{1/(\ast a-a+1)}. \tag{75}$$

Here $\theta = \cot^{-1}[aU/c_0]$, $a = (\gamma + 1)/[2n(\gamma - 1) + 4]$ and b is a constant. Thus we have

$$R = Ut = \frac{c_0}{ab} \cdot bt \cot \theta. \tag{76}$$

Since bt is given in terms of θ by (75), $R(t)$ is given parametrically in terms of θ. A graph of $[(ab)/c_0]R$ versus bt is given in Fig. 4 for $n = 3$ and $\gamma = 1.4$, which represents an expanding spherical shock in air. When the minus sign is chosen in (71)-(75) the same curves are obtained as with the plus sign, provided t is replaced by minus t, thus representing a converging shock. Of course, only one of any such pair of solutions can occur physically, since only one obeys the entropy inequality at the shock.

For small values of t we have

$$R \approx Kt^{2a} \tag{77}$$

where K is a constant. For $\gamma = 7$ and $n = 3$ we find $a = 1/5$ and thus $R \sim t^{2/5}$, which is the behavior of Primakoff's point blast solution for small t. For large t, R behaves linearly in t, as in (73).

To determine p_0 and F, we have from (69) and the condition $\gamma p_0\rho_0^{-1} = c_0^2$,

$$F = \frac{2\rho_0\dot{R}^2 - (\gamma - 1)p_0}{(\gamma + 1)j^{-n\gamma}} = \frac{2\gamma p_0}{(\gamma + 1)c_0^2}\dot{R}^2 j^{n\gamma} - \frac{\gamma - 1}{\gamma + 1} p_0 j^{n\gamma}. \tag{78}$$

Thus, if dot denotes time derivative and prime denotes derivative of a function with respect to its argument, we have

$$\begin{aligned} \dot{F} &= \frac{2\gamma}{(\gamma + 1)c_0^2}(p_0'\dot{R}^3 j^{n\gamma} + 2p_0\dot{R}\ddot{R} j^{n\gamma} + n\gamma p_0\dot{R}^2 j^{n\gamma-1} j') \\ &\quad - \frac{\gamma - 1}{\gamma + 1}(p_0'\dot{R} j^{n\gamma} + n\gamma p_0 j^{n\gamma-1} j') \\ &= F'(\dot{R} j^{-1} - Rj^{-2}j'). \end{aligned} \tag{79}$$

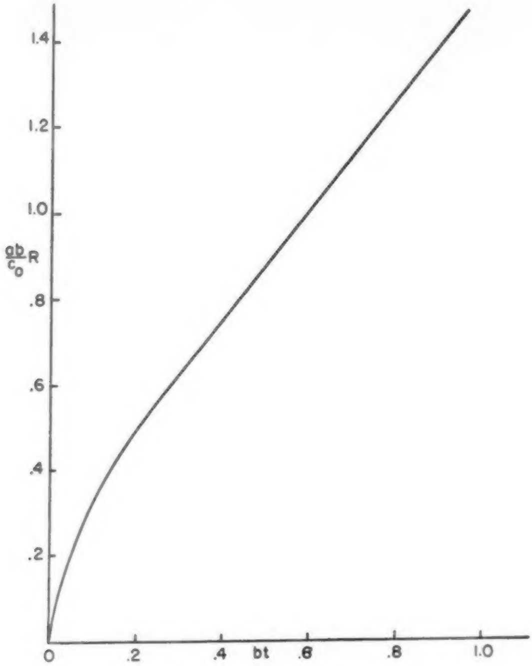


FIG. 4. The radius R of an expanding spherical shock in air as a function of time t computed from the parametric equations (75, 76) with $n = 3$, $\gamma = 1.4$.

Using (78) and (79) in (67) yields the following equation for p_0

$$\begin{aligned}\Bigg[\frac{2\gamma}{(\gamma+1)c_0^2}&(p_0'R^{\cdot 3}j^{n\gamma} + 2p_0R^{\cdot}R^{\cdot\cdot}j^{n\gamma} + n\gamma p_0R^{\cdot 2}j^{n\gamma-1}j') \\ &- \frac{\gamma-1}{\gamma+1}(p_0'R^{\cdot}j^{n\gamma} + n\gamma p_0 j^{n\gamma-1}j')\Bigg][-\lambda R j^{n-1}\gamma p_0 c_0^{-2}(R^{\cdot}j^{-1} - Rj^2j')]^{-1} \\ &= 2\gamma c_0^{-2}R^{\cdot 2}\left[\gamma + 1 + \frac{(\gamma-1)2\gamma R^{\cdot 2}}{(\gamma+1)c_0^2} - \frac{(\gamma-1)^2}{(\gamma+1)}\right]^{-1}. \qquad (80)\end{aligned}$$

Upon introducing $p_0^{\cdot} = p_0'R^{\cdot}$, the above equation can be integrated yielding $p_0[R(t)]$ as a function of t. To obtain $p_0(R)$, we must insert $t = t(R)$ from (75) and (76) into this result. Then when $p_0[R(t)]$ is known, F can be found from (78). These calculations will not be carried out here.

9. Conclusion and extensions. The problem of spherical, cylindrical or planar flow of a polytropic gas has been formulated in Lagrangian variables, giving rise to a non-linear second order partial differential equation in two variables, h and t. A class of special solutions of this equation has been found by separation of variables, and these solutions depend upon an arbitrary function which is related to the arbitrary entropy distribution in the gas. By specializing this function solutions corresponding to the expansion into a vacuum of an isentropic or non-isentropic gas cloud have been obtained

as well as solutions corresponding to the propagation of finite and of strong shocks in variable media.

Other types of gas motion can be described by solutions of other forms. For example, solutions of the form $y = f(\alpha t + \beta h)$, where α and β are constants, can easily be found, and they describe steady motions. Solutions of the progressing wave type, similar to those considered by G. Guderley, by G. I. Taylor, by J. von Neumann and J. Calkin, and by R. Courant and K. O. Friedrichs (*loc. cit.*, Chapter VI-C), can be obtained if y has the form

$$y = t^a f[K(h)t]. \tag{81}$$

Here a is a constant and $K(h)$ is a function of h to be determined. By inserting (81) into (8) we find that $K(h)$ must have the form

$$K(h) = (Bh + C)^{1/(1-\epsilon)}, \tag{82}$$

where B, C and ϵ are constants.

The resulting equation for f can be reduced to first order, following von Neumann and Calkin, by introducing the new variables s, $F(s)$ and $q(F)$ defined by

$$\begin{aligned} s &= \log\,[K(h)t], \\ F(s) &= e^{-\delta s} f(e^s), \\ q(F) &= \left(\frac{d}{ds} + \delta\right) F(s). \end{aligned} \tag{83}$$

In (83), the constant $\delta = [an(1 - \gamma) - 2a + 2][2 + n(\gamma - 1)]^{-1}$. The equation (8) for f now becomes the following equation for $q(F)$

$$\begin{aligned} &[1 - \gamma D F^{(n-1)(1-\gamma)} q^{-\gamma-1}](q - \delta F)q^{\cdot} \\ &= F^{(n-1)(1-\gamma)} q^{-\gamma} D[\gamma\epsilon + \gamma(n-1)qF^{-1} + \epsilon\gamma + a + 1 - \epsilon] - (a-1)aF - 2aq. \end{aligned} \tag{84}$$

In (84) D is a constant. Special cases of (84) have been studied by von Neumann and Calkin.

Note added in proof. The solution (24)–(26) for $\gamma \neq 1$ and $\lambda \neq 0$, when specialized to the case $g_0 = 0$, coincides with the result of L. I. Sedov, "On the integration of the equations of one-dimensional gas motion", Doklady, AN USSR **90**, 735 (1953). Some related results are given by M. L. Lidov, "Exact solution of the equations of one-dimensional unsteady gas motion taking into account Newtonian gravitational forces", Doklady, AN USSR **97**, 409–410 (1954).

EXPRESSION OF WAVE FUNCTIONS OVER A HALF SPACE IN TERMS OF THEIR BOUNDARY VALUES*

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1. Introduction. We consider the expression of a solution u of the wave equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{1.1}$$

over the half space $z > 0$, in terms of its boundary values over the boundary $z = 0$:

$$u(P, t) = u(x, y, z, t) = \frac{1}{2\pi}\left[\int [u]\, d\omega + \frac{1}{c}\left[\frac{\partial u}{\partial t}\right] R\, d\omega\right], \tag{1.2}$$

and the alternative expression of u over $z > 0$ in terms of the boundary values of $(\partial u/\partial z)$ given by:

$$u = -\frac{1}{2\pi}\int \left[\frac{\partial u}{\partial z}\right]\frac{dx'\, dy'}{R}. \tag{1.3}$$

Here the integration is carried out over the plane $z = 0$, $d\omega$ denoting the element of solid angle subtended at P by the plane element $dx'dy'$ at P', while

$$[u],\ [\partial u/\partial t],\ [\partial u/\partial z] \tag{1.4}$$

denote the retarded values of these functions at P', that is their values at P' not at the time t, but at the time $t' = t - R/c$, where R is the distance PP'. The "retarded time"

$$t' = t - R/c \tag{1.5}$$

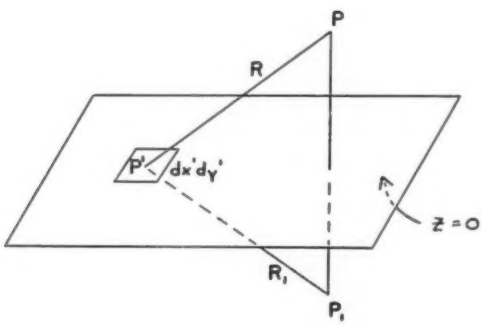

FIG. 1.

is the time t' such that a spherical wavelet, starting from P' at t' and with a radius expanding with a velocity c, would reach the point P at the time t, at which the left member of Eq. (1.2) is evaluated.

*Received Oct. 5, 1954; revised manuscript received August 24, 1955.

In the version of this paper first submitted to this journal, the authors derived several proofs of these equations, one based on Fourier integral time-resolution, another on a slight modification of the method used by Kirchhoff in expressing solutions of the non-homogeneous wave equation

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -4\pi\rho(x, y, z, t) \tag{1.6}$$

in terms of the "source" distribution ρ, as a sum of the retarded potentials of the latter, as follows (see for instance [1]):

$$u = \int \frac{[\rho]}{R}\, dv. \tag{1.7}$$

After this paper was revised in accordance with the recommendations of the referee, it was pointed out to the authors that Eq. (1.2), (1.3) appear in reference [2], where a hint regarding their proof is also given. It, therefore, seemed pertinent to omit proofs of (1.2), (1.3), and confine ourselves to several applications of these equations and their n-dimensional counterparts.

The boundary value expressions under discussion, it should be pointed out, differ from the more commonly considered solution of the Cauchy problem for (1.1), in which u is expressed for $t > 0$ in terms of the values of u, $\partial u/\partial t$ at the time $t = 0$.

It is to be emphasized that the expressions S_1 of the wave functions in terms of their boundary values considered here are unique *only* if, as is customary in physical applications, the *retarded* potentials are used. A similar, but *different* expression S_2, utilizing the *advanced* potentials also exists. Thus a solution of (1.6) exists of the form (1.7) but with $[\rho]/R$ replaced by the *advanced potential* $\rho(t + R/c)/R$. A linear combination of these solutions of the form $\lambda S_1 + (1 - \lambda)S_2$ for any constant λ also furnishes a possible solution.

In the special case where u is independent of time, Eq. (1.1) reduces to the Laplace equation, and Eqs. (1.2), (1.3) simplify to

$$u(P, t) = \frac{1}{2\pi}\int u\, d\omega, \tag{1.8}$$

$$u(P, t) = -\frac{1}{2\pi}\int \frac{\partial u}{\partial z}\frac{dx'\, dy'}{R}. \tag{1.9}$$

Both of these equations are familiar from potential theory.

2. Examples. A. *Axially symmetric product solutions of the wave equation.* As a first example we consider the axially symmetric wave function

$$u = J_0(\lambda r) \exp[-(\gamma^2 - k^2)^{1/2}z] \exp(i\alpha t), \qquad r^2 = x^2 + y^2, \tag{2.1}$$

for $\gamma > k$. Applying Eq. (1.3) for a point P on the positive z-axis, there results, upon introduction of polar coordinates in the plane of integration and cancellation of the factor $\exp(i\alpha t)$,

$$\frac{\exp[-(\gamma^2 - k^2)^{1/2}z]}{(\gamma^2 - k^2)^{1/2}} = \int_0^\infty \frac{\exp(-ikR)J_0(\gamma r)r\, dr}{R}. \tag{2.2}$$

Since (see Fig. 2)

$$R^2 = r^2 + z^2, \qquad R\, dR = r\, dr, \tag{2.3}$$

one obtains

$$\frac{\exp\,[-(\gamma^2 - k^2)^{1/2}z]}{(\gamma^2 - k^2)^{1/2}} = \int_z^\infty \exp\,(ikR) J_0[\gamma(R^2 - z^2)^{1/2}]\,dR. \tag{2.4}$$

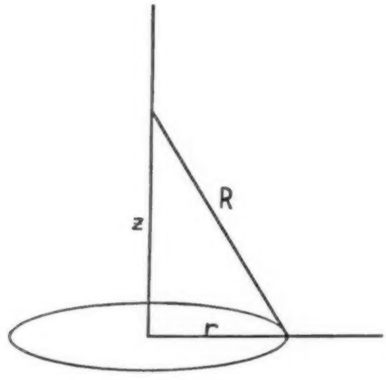

FIG. 2.

In a similar way one obtains for the same wave function (2.1), by applying either Eq. (1.2), the following integral

$$\begin{aligned}\exp\,[-(\gamma^2 - k^2)^{1/2}z] &= \int_0^\infty J_0(\gamma r) \exp\,(-ikR)(1 + ikR) z r R^{-3}\,dr \\ &= z \int_0^\infty \exp\,(-ikR)(1 + ikR) J_0[\gamma(R^2 - z^2)^{1/2}]\,dR/R^2.\end{aligned} \tag{2.5}$$

The integral (2.2) is essentially equivalent to (see Sommerfeld in [3], Watson [4])

$$\frac{\exp\,[ik(\gamma^2 + z^2)^{1/2}]}{(\gamma^2 + z^2)^{1/2}} = \int_0^\infty \lambda J_0(\lambda\gamma) \exp\,[-(\lambda^2 - k^2)^{1/2}z](\lambda^2 - k^2)^{-1/2}\,d\lambda. \tag{2.6}$$

Indeed, by carrying out the following substitutions in (2.6):

$$ik \to -z, \qquad \lambda \to r, \qquad r \to \gamma, \qquad z \to ik, \tag{2.7}$$

one transforms (2.6) into (2.5).

Further integrals can be obtained from the wave function (2.1) by choosing P in Eqs. (1.2), (1.3) not on the axis of symmetry. These integrals remain two-dimensional since the integrand is not axially symmetric about a line through P normal to the plane $z = 0$. However, by the application of the Neuman expansion the integral can be reduced to the axially symmetric case.

B. General axially symmetric wave functions. As our second example we consider any wave function $u(r, z, t)$ in $z > 0$ which is axially symmetric about the z-axis, that is, a solution of

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{2.8}$$

Eq. (1.3) yields, upon replacing r by $(R^2 - z^2)^{1/2}$, the following expression of u for a point on the positive z-axis in terms of $u_z = \partial u/\partial z$ along $z = 0$:

$$u(0, z, t) = -\int_z^\infty u_z[(R^2 - z^2)^{1/2}, 0, t - R/c]\, dR. \tag{2.9}$$

Eq. (1.2) yields similarly

$$\begin{aligned} u(0, z, t) = z \int_z^\infty u[(R^2 - z^2)^{1/2}, 0, t - R/c]R^{-1}\, dR \\ + (z/c) \int_z^\infty u[(R^2 - z^2)^{1/2}, 0, t - R/c]\, dR. \end{aligned} \tag{2.10}$$

C. Plane waves. As a third example consider an arbitrary *plane wave* solution of the form

$$u = f(t - z/c) \tag{2.11}$$

where f satisfies the conditions

$$\begin{aligned} f(s) &= 0 \quad \text{for} \quad s < 0, \\ f'(0) &= 0. \end{aligned} \tag{2.12}$$

Since the field is plane, it can be considered to be symmetric about the z-axis. Application of Eq. (1.3) for

$$t - z/c > 0, \qquad z > 0 \tag{2.13}$$

yields, upon introducing polar coordinates as in examples A and B,

$$\begin{aligned} f(t - z/c) &= -(1/c) \int_0^{r_1} f'(t - R/c)rR^{-1}\, dr \\ &= -(1/c) \int_z^{ct} f'(t - R/c)\, dR, \end{aligned} \tag{2.14}$$

where

$$r_1 = [(ct)^2 - z^2]^{1/2}.$$

That this is an identity follows immediately upon carrying out the integration.

Similar application of Eq. (1.2) to the plane wave function (2.11) satisfying (2.12) yields for z, t satisfying (2.13)

$$\begin{aligned} f(t - z/c) &= z \int_0^{r_1} [f(t - R/c)R^{-3} + f'(t - R/c)c^{-1}R^{-2}]r\, dr, \\ &= z \int_z^{ct} f(t - R/c)R^{-2}\, dR + zc^{-1} \int_0^{ct} f'(t - R/c)R^{-1}\, dR, \end{aligned} \tag{2.15}$$

where r_1 is as in (2.14). That this again is an identity follows by integrating the last integral by parts and utilizing (2.12).

It will be noted that the wave function (2.11) does *not* include a simple standing plane wave such as

$$u = \sin \alpha t \sin kz. \tag{2.16}$$

Indeed for this example the right-hand member of Eq. (1.2) vanishes and Eq. (1.2) obviously fails to hold. The function (2.16), if resolved into travelling waves, includes

a plane wave travelling in direction of *negative z*. The reader will discover in supplying the proof of theorem (1.2) that the integrals over $\sum$, a hemisphere of large radius R_2, which must converge to zero as $R_2 \to \infty$ to yield (1.2), fail to do so for the plane wave travelling in the direction of negative z.

D. Spherical wave. As our next example we consider a spherical wave originating from a point P_2, $z = -a$, $a > 0$, on the negative z-axis (see Fig. 3):

$$u = \frac{f(t - R_2/c)}{R_2}, \tag{2.17}$$

where R_2 is the distance from P_2. Again we assume that f satisfies the conditions (2.12).

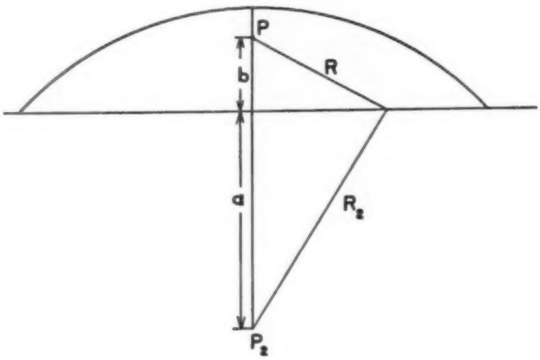

Fig. 3.

Eq. (1.2) now yields for a point P at $z = b$ on the positive z-axis for

$$t - (a + b)/c > 0, \tag{2.18}$$

upon recalling Eq. (2.10), the following relation

$$\frac{f[t - (a + b)/c]}{(a + b)} = b \int_b^{R'} \frac{f[t - (R_2 + R)/c]\, dR}{R_2 R^2} - \frac{b}{c} \int_b^{R'} \frac{f'[t - (R_2 + R)/c]\, dR}{R_2 R}. \tag{2.19}$$

The upper limit R' of the integrations is given by

$$R' + R_2 = ct. \tag{2.20}$$

Upon utilizing the relations

$$R^2 = b^2 + r^2, \qquad R_2^2 = a^2 + r^2 \tag{2.21}$$

one obtains from (2.19)

$$\frac{f[t - (a + b)/c]}{(a + b)} = b \int_a^{R_2'} f[t - (R_2 + R)/c][R_2^2 - (a^2 + b^2)]^{-3/2}\, dR_2 - (b/c) \int_a^{R_2'} f'[t - (R_2 + R)/c][R_2^2 - (a^2 + b^2)]\, dR_2, \tag{2.22}$$

where

$$R_2' = (a^2 - b^2)/2ct + ct/2. \tag{2.23}$$

That (2.22) is an identity follows by integrating the second integral by parts and utilizing the initial conditions (2.12).

3. Extension to n-dimensions. We now consider the extension of the above results to n-dimensional (Euclidean) spaces E_n and to solutions of the "wave equation"

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{3.1}$$

The aim is to express u over the region $x_n > 0$ in terms of its boundary values over $x_n = 0$.

The situation is quite different depending upon whether n is odd or even. For *odd* n an extension of Eq. (1.2) *is* possible and $u(P, t)$ can be expressed in terms of retarded values of u, $\partial u/\partial t$, $\cdots$, $\partial^{(n+1)/2} u/\partial t^{(n+1)/2}$ along the flat $x_n = 0$ as follows:

$$u(P, t) = \frac{2}{\Omega_n} \int \sum_{m=0}^{(n-1)/2} \frac{C_{nm} R^m}{c^m} \left[\frac{\partial^m u}{\partial t^m}\right] d\omega_n . \tag{3.2}$$

Here C_{nm} ; $m = 0, 1, \cdots$, are proper constants, the integration is carried out along $x_n = 0$, $d\omega_n$ denoting the element of "solid angle" subtended by $dx_1 \cdots dx_{n-1}$ at P' in $x_n = 0$ at the point P:

$$d\omega_n = \frac{x_n}{R^n}\, dx_1' \cdots dx_{n-1}' , \tag{3.3}$$

while $[u]$, $[\partial u/\partial t]$, $\cdots$, as for $n = 3$, are the values of u at P' at the time $t - R/c$, where $R = PP'$; finally

$$\Omega_n = \int d\omega_n = n[\Gamma(1/2)]^n/\Gamma[(n/2) + 1] \tag{3.4}$$

is the complete solid angle, that is the $(n - 1)$-dimensional content of the "surface" of the n-dimensional unit sphere:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \tag{3.5}$$

Similarly, for odd n, a generalization of Eq. (1.3) is possible, and is given by

$$u(P, t) = \frac{2}{\Omega_n} \int \cdots \int \sum_{m=0}^{(n-3)/2} \frac{R^m D_{nm}}{c^m} \left[\frac{\partial^{m+1} u}{\partial x_n' \,\partial t^m}\right] dx_1' \cdots dx_{n-1}' , \tag{3.6}$$

where again D_{nm} are proper constants.

For *even* n the situation is quite different, and the expression of u in terms of its boundary values is more complicated. This is but one instance of the many differences between solutions of the wave equation in even and odd number of dimensions. In the present case, one may ascribe this difference to the fact that the Bessel functions which are involved in the proof of Eqs. (3.2) or (3.6) turn out to be of integral order for even n, but of an order that differs from an integer by a half for odd n. The latter Bessel functions, as is well known, can be expressed in terms of exponentials and a product of a finite number of negative fractional powers of the argument; no such simple expression, however, is possible for the Bessel functions of integral order.

The expression of u in terms of its boundary values for even n is considered briefly toward the end of this section, and in more detail in the following section.

After these preliminaries we indicate briefly the generalizations of (1.2), (1.3) for any number of dimensions n by a method similar to the Fourier integral method mentioned in Sec. 1. We resolve u, the solution of (3.1), into a Fourier integral in time:

$$u = \int_{-\infty}^{\infty} \exp (i\alpha t) U(x_1, x_2, \cdots, x_n) \, d\alpha, \tag{3.7}$$

where (it is assumed that) the integrand satisfies Eq. (3.1), and hence U is a solution of the n-dimensional equation

$$\nabla^2 U + k^2 U = 0, \qquad k = \alpha/c. \tag{3.8}$$

We apply Green's theorem to U and the Green's function G for the half space $x_n > 0$ and the condition $U = 0$ on $x_n = 0$:

$$G(P, P') = f_n(k, R) - f_n(k, R_1), \tag{3.9}$$

where R is the distance from P, R_1 the distance from its mirror image in $x_n = 0$, while $f_n(k, R)$ is a proper, spherically symmetric solution of (3.8), which behaves like

$$\text{Const. } \exp (ikR)/R^{(n-1)/2} \tag{3.10}$$

at infinity and like

$$\begin{cases} R^{2-n}/(n-2) & \text{for} \quad n > 2, \\ -\ln R & \text{for} \quad n = 2, \end{cases} \tag{3.11}$$

near $R = 0$. There results

$$U(P) = \frac{2}{\Omega_n} \int \cdots \int x_n U(x_1, x_2, \cdots, x_{n-1}, 0) \{\partial f_n(k, R)/R \, \partial R\} \, dx_1' \cdots dx_{n-1}', \tag{3.12}$$

and, upon utilization of Eq. (2.3),

$$U(P) = \frac{2}{\Omega_n} \int \cdots \int U(x_1, x_2, \cdots, x_{n-1}, 0) R^{n-1} \, \partial f_n(k, R)/\partial R \, d\omega_n . \tag{3.13}$$

In a similar way, by replacing G by the sum of the two right-hand terms in Eq. (3.9), one obtains

$$U(P) = 2 \int \cdots \int U_{x_n}(x_1, x_2, \cdots, x_{n-1}, 0) f_n(k, R) \, dx_1' \cdots dx_{n-1}' . \tag{3.14}$$

To obtain f_n we note that for spherically symmetric solutions Eq. (3.8) reduces to

$$\frac{\partial^2 U}{\partial R^2} + \frac{n-1}{R} \frac{\partial U}{\partial R} + k^2 U = 0. \tag{3.15}$$

Solutions of this equation are given by

$$U = R^{1-n/2} B_{n/2-1}(kR) \tag{3.16}$$

where B_ν is a Bessel function of order ν. We shall choose

$$f_n = \text{Const } R^{1-n/2} H^{(2)}_{n/2-1}(kR), \tag{3.17}$$

since the asymptotic form of this solution yields a behavior at infinity of the form (3.10) which is more in accord with a "retarded potential" solution corresponding to an expanding spherical wave, than would be the case with any other Bessel function.

The difference between even and odd n appears when we examine the expansions for $H_{n/2-1}$ (see [3], p. 198, Eq. (6)):

$$H_\nu^{(2)}(z) \cong \left(\frac{2}{\pi z}\right)^{1/2} \exp\left[-i(z - \nu\pi/2 - \pi/4)\right] \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2iz)^m}, \qquad (\nu, m) = \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2) \cdots [4\nu^2 - (2m-1)^2]}{2^{2m} m!}. \tag{3.18}$$

For general ν these expansions are asymptotic, and divergent. However, if $\nu = n/2 - 1$ where n is an odd integer, then $(\nu, m) = 0$ for $m \geqq (n-1)/2$, the series in (3.18) reduces to $(n-1)/2$ terms, and yields an exact representation for $H_\nu^{(2)}$.

Hence for odd n, one obtains from Eqs. (3.16), (3.17), upon adjusting the multiplicative constant in accordance with Eq. (3.11),

$$\begin{aligned} f_n(k, R) &= \frac{R^{2-n} \exp(-ikR)}{(n-2)[(n/2) - 1, (n-3)/2]} \sum_{m=0}^{(n-3)/2} ((n/2) - 1, m)(2ikR)^{(n-3)/2-m} \\ &= \exp(ikR)\, R^{2-n} \sum_{m=0}^{(n-3)/2} D_{nm}(ikR)^m. \end{aligned} \tag{3.19}$$

Substituting this expansion in Eq. (3.13) and recalling the factor $\exp(i\alpha t)$ one obtains

$$\begin{aligned} &U(P) \exp(i\alpha t) \\ &= -\frac{2}{\Omega_n} \int \cdots \int U_{x_n}(x_1', x_2', \cdots, x_{n-1}', 0) \sum_{m=0}^{(n-3)/2} \exp\left[i(\alpha t - kR)\right] D_{nm}(ikR)^m. \end{aligned} \tag{3.20}$$

Now replace the left-hand member by $u(P, t)$; note that multiplication by the factor ik is equivalent to application of the operator $(1/c)(\partial/\partial t)$, and that t occurs only in the form $\exp[i\alpha(t - R/c)]$. Hence for functions u of the form $\exp(i\alpha t)\, U(x_1 \cdots x_n)$ Eq. (3.20) may be recast in the form (3.6).

For general wave functions, Eq. (3.6) now follows by means of Fourier integral superposition, as in Eq. (3.7).

A similar proof of Eq. (3.2) follows from Eq. (3.12). By differentiation of (3.19) one obtains

$$\frac{\partial f_n}{\partial R} = \exp(ik\,R)\, R^{1-n} \sum_{m=0}^{(n-1)/2} C_{nm}(ikR)^m, \tag{3.21}$$

where

$$\begin{aligned} C_{n0} &= (2 - n) D_{n0}, \\ C_{nm} &= ikD_{n,m-1} + (m - n)D_{n,m}, \qquad m > 0. \end{aligned} \tag{3.22}$$

The case $n = 3$ has been considered in Sec. 1. For this case the series in Eq. (3.18) reduces to its first term, unity, and Eq. (3.19) yields

$$f_3 = \exp(-ikR)/R. \tag{3.23}$$

For $n = 5, 7$ one obtains

$$f_5(k, R) = \frac{\exp(ikR)}{3R^3}[1 + ikR], \tag{3.24}$$

$$f_7(k, r) = \frac{\exp(-ikR)}{5R^5}\left[1 + \frac{ikR}{2} + \frac{(ikR)^2}{6}\right]. \tag{3.25}$$

It is of interest to note that for *odd* n the functions (3.19), multiplied by the factor $\exp(i\alpha t)$, can be written in the form

$$u_n(R, t) = R^{2-n} \sum_{m=0}^{(n-3)/2} \frac{R^m D_{nm}}{c^m} \frac{\partial^m}{\partial t^m} F\left(t - \frac{R}{c}\right) \tag{3.26}$$

where

$$F(u) = \exp(i\alpha u). \tag{3.27}$$

Hence, by superposition over α, it follows that (3.26) is a solution of the wave equation (3.1) for an *arbitrary* function F. This is the general expanding spherical wave, and is analogous to $F(t - R/c)/R$ for $n = 3$, to which, indeed, it reduces for $n = 3$.

Returning to the case of even n, now one can no longer utilize the asymptotic Eq. (3.18) to advantage for the integral order Bessel functions. Equations (3.17), (3.11) yield (for both even and odd n)

$$f_n = \begin{cases} \pi i H_0(\alpha R/c) \quad \text{for} \quad n = 2, \\ \dfrac{\pi i}{(n-2)\left(\dfrac{n}{2} - 2\right)!} \left(\dfrac{\alpha}{2c}\right)^{(n/2-1)} R^{1-n/2} H_{(n/2-1)}(\alpha R/c) \quad \text{for} \quad n > 2. \end{cases} \tag{3.28}$$

Again, Eqs. (3.13), (3.14) can be used, combined with Fourier α-integration as in Eq. (3.7), to yield the desired expression of u in terms of its boundary values, but the result is rather complex. More attractive forms are considered in the following section.

4. Alternative methods in n dimensions. With the odd-dimensional case disposed of, one may treat the case of even n by immersing the space E_n of $(x_1, x_2, \cdots, x_n)$ in an odd-dimension space E_{n+1} obtained by adding one coordinate x_{n+1} to E_n, and regarding the solution u of Eq. (3.1) as a solution in E_{n+1} of the corresponding $(n + 1)$-dimensional wave equation. This is sometimes known as the "principle of descent" (see [5]).

As a first application of this "principle" we shall obtain for even n a general expression for $u_n(R, t)$, and expanding spherical wave. This will be done by immersing E_n in E_{n+1}, lining the x_{n+1}-axis with a uniform distribution per unit x_{n+1} of $(n + 1)$-dimensional point "sources", and adding the resulting wave functions given by Eq. (3.26) with n replaced by $(n + 1)$, for the elements of the wave components. Thus, for $n = 2$, we immerse the (x, y)-plane in an (x, y, z)-space, and obtain a wave of cylindrical symmetry by adding spherical wave elements originating from sources distributed uniformly over the z-axis.

For general even n the resulting wave function is independent of x_{n+1} and can be evaluated in the flat $x_{n+1} = 0$. There results

$$u_n(R, t) = 2 \int_{-\infty}^{+\infty} \sum_{m=0}^{n/2-1} (R'')^{1+m-n} D_{n+1,m} c^{-m} F^{(m)}(t - R''/c)\, dx_{n+1}, \tag{4.1}$$

where

$$R'' = [R^2 + (x_{n+1})^2]^{1/2}. \tag{4.2}$$

Changing to $s = R''$ as a variable of integration, one obtains

$$u_n(R, t) = 2 \int_R^\infty \sum_m \frac{s^{2+m-n} D_{n+1,m}}{c^m(s^2 - R^2)^{1/2}} F^{(m)}(t - s/c)\, ds; \tag{4.3}$$

while introduction of

$$x_{n+1} = R \sinh v, \qquad s = R'' = R \cosh v, \tag{4.4}$$

yields

$$u_n(R, t) = \int_{-\infty}^{+\infty} \sum_m D_{n+1,m}(R \cosh v)^{2+m-n} c^{-m} F^{(m)}[t - (R/c) \cosh v]\, dv. \tag{4.5}$$

For $n = 2$, Fig. 4 shows in perspective the (x, y)-plane, its "immersion" in the 3-space E_3, and the various distance R, R'', $dx_3 = dz$ involved in Eqs. (4.1), (4.2).

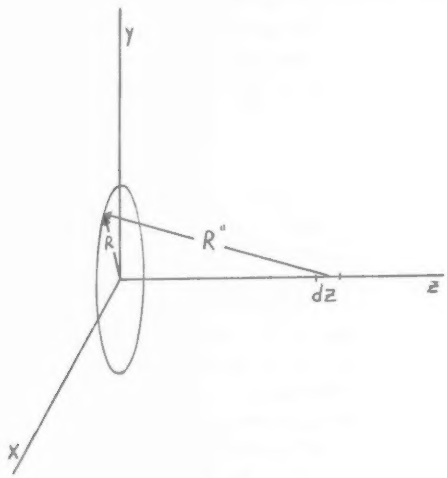

FIG. 4

If F satisfies the conditions

$$\begin{aligned} F(s) &= 0 \quad \text{for} \quad s \leqq 0, \\ f(0) &= F'(0) = \cdots F^{(n/2-1)}(0), \end{aligned} \tag{4.6}$$

the upper limits in the integrations in Eqs. (4.3), (4.5) are replaced by ct, $\cosh^{-1}(ct/R)$. For such a case, for $n = 2$, Fig. 5 shows for a fixed t, both in perspective and in the plane $z = 0$, the spherical wave fronts corresponding to the argument 0 for F.

For $n = 2, 4$, (with (4.6) holding), Eqs. (4.1)-(4.5) yield

$$\begin{aligned} u_2(R, t) &= 2 \int_R^{ct} F(t - s/c)(s^2 - R^2)^{-1/2} ds \\ &= 2 \int_0^{\cosh^{-1}(ct/R)} F[t - (R/c) \cosh v]\, dv; \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
u_4(R, t) &= \frac{2}{3}\int_R^{ct}\left[\frac{F(t - s/c)}{s^2} - \frac{F'(t - s/c)}{sc}\right]\frac{ds}{(s^2 - R^2)^{1/2}} \\
&= \frac{2}{3}\int_0^{\cosh^{-1}(ct/R)}\left[\frac{F(t - (R/c)\cosh v)}{\cosh^2 v} - \frac{RF'(t - (R/c)\cosh v)}{\cosh v}\right] dv.
\end{aligned} \tag{4.8}
$$

It is possible by integration by parts to eliminate the derivatives F', F'', from Eqs. (4.3), (4.5) leaving only F in the integrand.

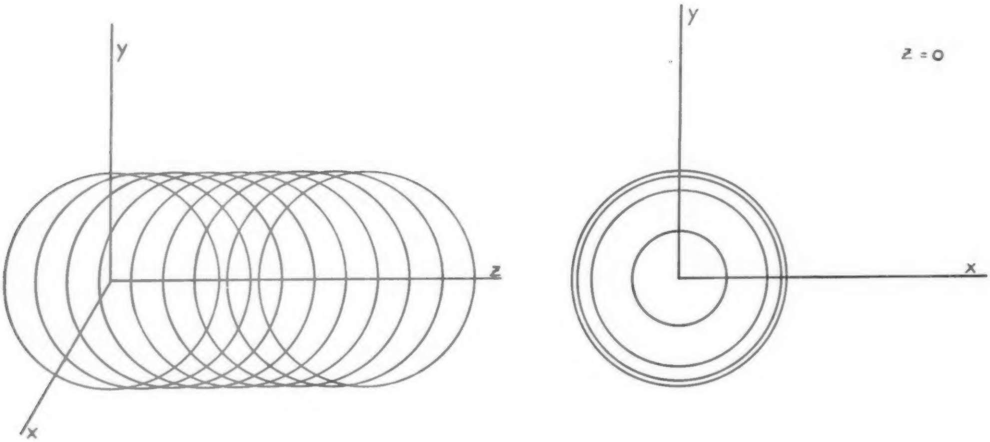

FIG. 5.

It will be noticed from Fig. 5 and Eqs. (4.1)-(4.8) that if F satisfies the conditions (3.2), the wave function $u_n(R, t)$ does not vanish outside the spherical shell

$$0 < R/c - t < \delta. \tag{4.9}$$

In this respect the case of even n differs from the odd one.

Of special interest, both for odd and even n, is the limiting wave function $V_n(R, t)$ approached by $u_n(R, t)$ by letting F above approach a proper constant multiple of the Dirac δ-function:

$$F(t) \to C\delta(t). \tag{4.10}$$

The function $V_n(R, t)$ may be regarded as the retarded potential solution due to pulse point source at the origin, at time $t = 0$, that is, it is the "retarded" solution of

$$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\Omega_n \delta(x_1, x_2, \cdots, x_n, t), \tag{4.11}$$

where

$$\delta(x_1 \cdots t) = \delta(x_1)\,\delta(x_2)\cdots\delta(t). \tag{4.12}$$

The functions V_n for n even or odd, are useful in deriving the following n-dimensional analogue of Kirchhoff's theorem. The solution of wave equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\Omega_n \rho(x_1, \cdots, x_n, t) \tag{4.13}$$

with a "source" P, is given by

$$u(P, t) = \int_{-\infty}^{t} dt' \int dP' \rho(P', t') V_n(R, t - t')\, dP' \tag{4.14}$$

where

$$R = PP'. \tag{4.15}$$

For odd n, recalling Eqs. (3.26), (4.10) and replacing $F(t)$ by $\delta(t)$, one may convert the right-hand member of Eq. (4.14) to a form resembling Eq. (1.7); for even n, however, the existence of the "tail" in the wave element originating from each source element prevents a true retarded potential solution from ever acquiring a form similar to Eq. (1.7).

Finally, the function $V_n(R, t)$ can be used to obtain analogues of Eqs. (1.2), (1.3) for n-dimensions. This will be illustrated for $n = 2$. Equations (4.7) now yield

$$V_2(R, t) = \begin{cases} [(ct)^2 - R^2]^{-1/2} & \text{for} \quad R < ct, \\ 0 & \text{for} \quad R < ct. \end{cases} \tag{4.16}$$

By utilizing this function, one obtains the following expression in $y > 0$ of the solution u of

$$\frac{\partial^2 u}{\partial x^2} + \frac{p^2 u}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \tag{4.17}$$

in terms of its values along $y = 0$, along the lines of Eqs. (1.2), (1.3):

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{t} dt' \int_{-\infty}^{+\infty} dx'[u(x', 0, t')y\ V_2(R, t - t')/R\ \partial R], \tag{4.18}$$

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{t} dt' \int_{-\infty}^{+\infty} dx'[u_y(x', 0, t')V_2(R, t - t')], \tag{4.19}$$

where

$$R^2 = (x - x')^2 + y^2.$$

An alternative way of arriving at the expression of the two-dimensional wave function $u(x, y)$ in $y > 0$ in terms of its values on $y = 0$, is again by applying the "method of descent", by considering u as a wave function in three dimensions, and applying Eqs. (1.1), (1.3) to the half space $y > 0$. First one obtains, since u is independent of z,

$$u(x, y, t) = (1/2\pi) \iint [u(x', 0, t - R''/c)\, y/(R'')^3$$
$$+ u_t(x', 0, t - R''/c)\, y/cR'']\, dx'\, dz', \tag{4.20}$$

$$u(x, y, t) = -\frac{1}{2\pi}\iint [u_y(x', 0, t - R''/c)/R'']\, dx'\, dz'$$

where

$$(R'')^2 = (x - x')^2 + y^2 + (z')^2. \tag{4.21}$$

By a proper change of variables (and integration by parts) it is possible to convert the z'-integration of Eqs. (4.20), (4.21) into a t'-integration as in Eqs. (4.18), (4.19), and derive the latter equations from the former.

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—NOTES—

CONCERNING THE EIGENVALUES OF A DIFFERENTIAL EQUATION IN CONVECTIVE HEAT TRANSFER*

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In a recent paper by Batchelor [1] there occurs the differential equation

$$d^2f/dy^2 + \lambda y(\tfrac{1}{2} - y)f = 0. \tag{1}$$

It is required to determine the values of λ for which (1) possesses a non-trivial solution, subject to the conditions

$$f(0) = f(\tfrac{1}{2}) = 0. \tag{2}$$

An approximate value of λ was obtained in the above cited paper by expanding "f" in a power series in y. Unfortunately, such an expansion is too slowly convergent to give a good estimate of λ with only a limited number of terms. However, if the function f is expanded as a power series in the eigenvalue λ, a much better value can be obtained with an equivalent number of terms. The means of obtaining such a series is quite simple and in fact a well known method, although its application to eigenvalue problems does not appear to have received the attention which would seem warranted in view of its utility as a means of obtaining numerical results.

A more convenient form of the equation for this method is obtained by setting $y = (t + 1)/4$. Denoting derivatives with respect to "t" by primes, the new equation is

$$f'' + \alpha^2(1 - t^2)f = 0, \tag{3}$$

where $\alpha^2 = \lambda/(16)^2$, and the boundary conditions are $f(\pm 1) = 0$. It may be noted that the general solution of Eq. (3) may be given in terms of the confluent hypergeometric function [2]. However, because of the limited tabulation of this function, this form of the solution is of little value in the present eigenvalue problem. Therefore, proceeding according to the previously mentioned plan of obtaining a formal expansion as a power series in α^2, we assume that $f(t)$ may be written in the form

$$f(t) = f_0(t) + \alpha^2 f_1(t) + \alpha^4 f_2(t) + \cdots + . \tag{4}$$

If this series is inserted in the differential equation (3), and the coefficients of the various powers of α^2 set equal to zero, there results the following series of equations:

$$f_0''(t) = 0, \qquad f_1''(t) = (t^2 - 1)f_0\,, \qquad f_2''(t) = (t^2 - 1)f_1\,, \qquad \text{etc.} \tag{5}$$

Solving this system in succession gives the desired coefficients in (8) as functions of "t". The boundary conditions require that $f(1) = 0$, which results in the characteristic equation in the form

$$f_0(1) + \alpha^2 f_1(1) + \alpha^4 f_2(1) + \cdots + = 0. \tag{6}$$

*Received Feb. 21, 1955; Revised manuscript received March 30, 1955.

If the series is terminated with the nth term, then (6) is merely a polynomial equation in α^2 whose roots are approximates to the eigenvalues.

The solutions of (5) for the case where f is even in t are given below

$$f_0(t) = 1,$$

$$f_1(t) = (-.5t^2 + .8333333t^4) \times 10^{-1},$$

$$f_2(t) = (4.1666667t^4 - 1.9444444t^6 + .1488095t^8) \times 10^{-2},$$

$$f_3(t) = (-1.3888889t^6 + 1.0912697t^8 - .2325837t^{10} + .0112735t^{12}) \times 10^{-3},$$

$$f_4(t) = (.2480158t^8 - .2755732t^{10} + .1002917t^{12} - .0133987t^{14} + .0004697t^{16}) \times 10^{-4},$$

$$f_5(t) = (-.275573t^{10} + .0412367t^{12} - .0206519t^{14} + .0047371t^{16} - .0004532t^{18} + .0000124t^{20}) \times 10^{-5},$$

$$f_6(t) = (.0020877t^{12} - .0037799t^{14} + .0025787^{16} - .0008297t^{18} + .0001366t^{20} - .0000101t^{22} + .00000002t^{24}) \times 10^{-6}.$$

Hence

$$f_0(1) = 1,$$

$$f_1(1) = -4.166667 \times 10^{-1}, \quad f_2(1) = 2.3710318 \times 10^{-2}, \quad f_3(1) = -.5189294 \times 10^{-3},$$

$$f_4(1) = .0598053 \times 10^{-4}, \quad f_5(1) = -.0042472 \times 10^{-5}, \quad f_6(1) = .0002044 \times 10^{-6}.$$

If only terms through α^6 are retained the equation

$$\beta^3 - 4.1666667\beta^2 + 2.3710318\beta - .5189294 = 0 \tag{7}$$

is obtained, where for convenience we have set $10/\alpha^2 = \beta$. The largest root of (7) is

$$\beta_1 = 3.528,$$

and the corresponding values of α^2 and λ are

$$\alpha_1^2 = 2.834, \qquad \lambda_1 = 725.6.$$

Listed below are the corresponding results obtained when 5, 6, and 7 terms are carried in the series:

Five terms:

$$\beta^4 - 4.1666667\beta^3 + 2.3710318\beta^2 - .5189294\beta + .0528053 = 0,$$
$$\beta_1 = 3.53638, \qquad \alpha_1^2 = 2.82775, \qquad \lambda_1 = 723.9. \tag{8}$$

Six terms:

$$\beta^5 - 4.1666667\beta^4 + 2.3710318\beta^3 - .5189294\beta^2 + .0598053\beta - .0042472 = 0,$$
$$\beta_1 = 3.536265, \qquad \alpha_1^2 = 2.827762, \qquad \lambda_1 = 723.91. \tag{9}$$

Seven terms:

$$\beta^6 - 4.1666667\beta^5 + 2.3710318\beta^4 - .5189294\beta^3 + .0598053\beta^2 - .0042472 + .0002046 = 0,$$
$$\beta_1 = 3.5363645, \qquad \alpha_1^2 = 2.827763, \qquad \lambda_1 = 723.907. \tag{10}$$

From the above results it may be safely concluded that the first eigenvalue of Eq. (1) is $\lambda_1 = 723.907$, correct to at least six figures.

It is theoretically possible to obtain not only the first, but also the higher, eigenvalues by this method. However, meaningful results can only be obtained for the higher ones if a very high degree of accuracy in the coefficients exists. With the present figures, the second highest roots of Eqs. (9) and (10) are respectively, .276 and .313. The latter value is probably correct to at least two places; the corresponding value of α_2^2 is 32.0.

It may be added that this same equation has been treated by Purday [3], also using a series in the independent variable. The results obtained there are

$$\alpha_1^2 = 2.83, \qquad \alpha_2^2 = 32$$

which agree with the values of the present analysis.

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A USEFUL INTEGRAL FORMULA FOR THE INITIAL REDUCTION OF THE TRANSPORT EQUATION*

By RICHARD L. LIBOFF (*Army Chemical Center*)

A theorem pertaining to spherical harmonics** states that if an angle β is determined by two directions, whose coordinates on the surface of a unit sphere are (θ, α) and (θ', α'), and $F(\beta)$ is a function expandable in a series of spherical harmonics in argument $(\cos \beta)$,

$$F(\beta) = \sum_l \frac{2l+1}{4\pi} k_l P_l(\cos \beta), \tag{1}$$

then

$$\int_{4\pi} F(\beta) Y_l(\theta, \alpha)\, d\omega = k_l Y_l(\theta', \alpha'). \tag{2}$$

$Y(\theta, \alpha)$ may either be spherical surface harmonic, a tesseral harmonic, or a spherical harmonic.

The theorem has been noted because of its applicability in diffusion theory, for problems which militate use of the transport equation.

The transport equation for various types of scattering (e.g., neutron, gamma, light) may in general, be written,

$$\frac{1}{\mu} \mathbf{s} \cdot \nabla N(\mathbf{r}, \mathbf{s}) = -N + \frac{1}{4\pi} \int_{4\pi} p(\beta) N(\mathbf{r}, \mathbf{s}')\, d\omega' + S. \tag{3}$$

*Received April 13, 1955. The topics of this paper appear in greater detail, in a report soon to be published by these Laboratories. (CRLR #479).

**Hobson, *Spherical and Ellipsoidal Harmonics*, Cambridge, 1931, p. 146.

Equation (3) states that the spatial rate of change of the distribution function $N(\mathbf{r}, \mathbf{s})$ at the position $\mathbf{r}$, in the direction of the unit vector $\mathbf{s}$, is due to three effects which appear on the right side of the equation. They are, in order of their appearance, first, radiation lost through absorption, and scattering out of the direction $\mathbf{s}$; secondly radiation which is scattered from all 4π solid angle into the direction $\mathbf{s}$. $p(\beta)$ is the angular differential cross section for the specific process involved. Thirdly there is the source contribution S. It is assumed in the above equation that a scatter is not accompanied by a change in wave length. μ is the total narrow beam absorption coefficient.

When the solution of (3) is through expansion processes, Eq. (2) becomes a useful integral formula for reducing the integral of (3) to its equivalent summation.

For instance, consider the problem which permits the unit vector $\mathbf{s}$ to be replaced by the ordinate angle θ (e.g., spherical symmetry and infinite plane source). In this case the distribution function may be expanded as

$$N(\mathbf{r}, \mathbf{s}') = \sum_l \frac{2l+1}{4\pi} a_l(\mathbf{r}) P_l(\cos\theta'). \tag{4}$$

Substitution of this expansion into the kernel of (3) will transform the original integral into a series of integrals, each term of which is similar to expression (2). From this we may write for the integral of (3),

$$\sum_l \frac{2l+1}{4\pi} a_l(\mathbf{r}) k_l P_l(\cos\theta), \tag{5}$$

k_l is the Legendre coefficient of the expansion of $p(\beta)$.

Following this with the replacement of $N(\mathbf{r}, \mathbf{s})$ by its equivalent expansion (4) into (3), will reduce the original integro-differential transport equation to a system of differential equations involving the sequence $\{a_l(\mathbf{r})\}$, knowledge of which completely determines $N(\mathbf{r}, \mathbf{s})$.

For more complicated geometries, where expansions in tesseral, or spherical surface harmonics are called for, the formula may be used in like manner, reducing the integral term of the transport equation to its corresponding summation, whence usually, a reduced system of equations is easily derivable.

TWO REMARKS ON HEISENBERG'S THEORY OF ISOTROPIC TURBULENCE*

By WILLIAM H. REID[1] (*Trinity College, Cambridge, England*)

1. Introduction. The exact dynamical equation for the rate of change with time of the energy spectrum function $E(k)$ in isotropic turbulence may be written in the form

$$\partial E(k)/\partial t = T(k) - 2\nu k^2 E(k), \tag{1}$$

where $T(k)$ is the transfer function usually denoted by this symbol. The incompleteness of Eq. (1) is well known and, in the past, several so-called "physical transfer theories" have been proposed in which a further relationship between $T(k)$ and $E(k)$ is postulated;

*Received May 9, 1955.

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the specific form of the relationship then follows from the particular mechanism of energy-transfer considered and from certain general dimensional considerations.

The theory which has attracted the greatest attention is the one due to Heisenberg [3] and is based on the concept of an eddy viscosity; in this theory $T(k)$ and $E(k)$ are related by an equation which may be written in the form

$$T(k) = -2\kappa \frac{d}{dk} \int_k^\infty [E(k')/k'^3]^{1/2} \, dk' \int_0^k k''^2 E(k'') \, dk''. \tag{2}$$

In this equation, κ is, by hypothesis, a constant of order unity. A number of writers have attempted to derive the value of κ under widely differing conditions and, in particular, existing results show a large variation of κ with the Reynolds number of the turbulence.

One of the purposes of the present note, therefore, is to show that these results are partially in error and that in fact κ, or more precisely the ratio S/κ, where S is the skewness factor of $-\partial u_1/\partial x_1$ is practically independent of the Reynolds number and thus to remove one cause for criticism of Heisenberg's theory.

2. The behavior of S/κ for small values of the Reynolds number. Since S is directly related to the second moment of the transfer function, one must first derive the explicit solution for $T(k)$. For small values of the Reynolds number, this can easily be done by expressing the solution as a power series in the Reynolds number. Thus, by assuming the usual type of initial period similarity

$$E(k, t) = \kappa^{-2} \nu^{3/2} t^{-1/2} F(x) \quad \text{and} \quad T(k, t) = \kappa^{-2} \nu^{3/2} t^{-3/2} U(x), \tag{3}$$

where $x = (\nu k^2 t)^{1/2}$, Eqs. (1) and (2) become

$$xF'(x) + (4x^2 - 1)F(x) = 2U(x) \tag{4}$$

and

$$U(x) = -2 \frac{d}{dx} \int_x^\infty [F(x')/x'^3]^{1/2} \, dx' \int_0^x x''^2 F(x'') \, dx''. \tag{5}$$

From Eq. (5) it is clear that for small values of the Reynolds number $U(x)$ is of higher order that $F(x)$ and since R_λ, the Reynolds number of the turbulence usually denoted by this symbol, always occurs in the combination κR_λ, we may write

$$F(x) = \sum_{n=2}^\infty F_n(x)(\kappa R_\lambda)^n \quad \text{and} \quad U(x) = \sum_{n=3}^\infty U_n(x)(\kappa R_\lambda)^n. \tag{6}$$

By substituting these series into Eqs. (4) and (5) and equating like powers of κR_λ, one obtains a sequence of equations for the determination of the functions $F_n(x)$ and $U_n(x)$. In particular, the first approximation to each of these functions can be readily obtained in the form

$$F_2(x) = \tfrac{3}{5} x e^{-2x^2} \tag{7}$$

and

$$U_3(x) = \frac{1}{8} \left(\frac{3}{5}\right)^{3/2} \frac{d}{dx} \mathrm{Ei}\,(-x^2)[1 - (1 + 2x^2)e^{-2x^2}], \tag{8}$$

where $-\mathrm{Ei}(-x)$ is the exponential integral function defined by

$$-\mathrm{Ei}\,(-x) = \int_x^\infty \frac{e^{-t}}{t} \, dt.$$

These functions are shown graphically in Fig. 1. In passing, it may be noted that $F_2(x)$, unlike $U_3(x)$, is independent of the assumed form of the transfer theory.

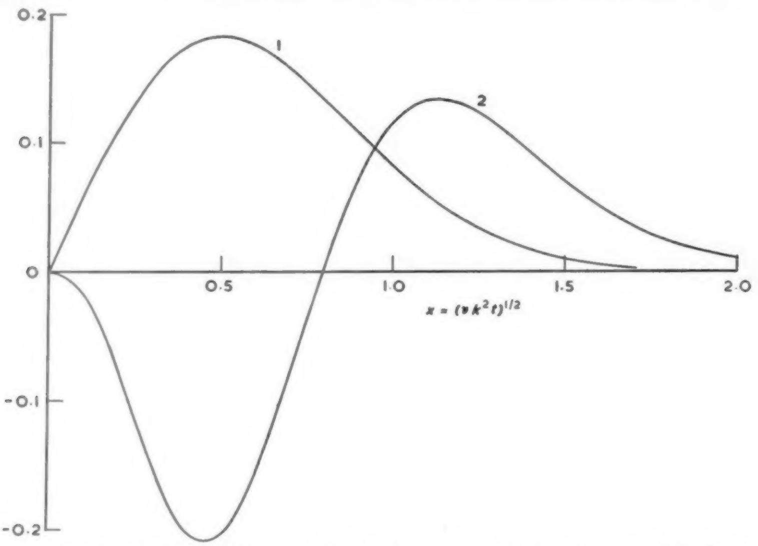

FIG. 1. Curve 1: The 'zero Reynolds number' energy spectrum, $F_2(x)$; independent of transfer theory. Curve 2: The first approximation to the transfer function, $10U_3(x)$; Heisenberg's transfer theory.

For small values of the Reynolds number, the ratio S/κ is then given by

$$\frac{S}{\kappa} = \frac{400}{7} \int_0^\infty x^2 U_3(x)\, dx + O(\kappa R_\lambda). \tag{9}$$

For the transfer function (8), the integral occurring in this equation has the value

$$\int_0^\infty x^2 U_3(x)\, dx = \frac{3(15)^{1/2}}{200} \left(\frac{4}{3} - \log 3 \right)$$

and hence

$$S/\kappa \to 0.78 \qquad \text{as} \qquad R_\lambda \to 0. \tag{10}$$

Experimentally, we know that $S \doteqdot 0.48$ at the smallest Reynolds number for which experiments have been made [2, Fig. 6.3] and this makes $\kappa \doteqdot 0.62$. The numerical coefficient in Eq. (10) differs considerably, however, from the one given by Proudman [6]. Presumably, his result was obtained by expanding $U_3(x)$ in powers of x; in view of the somewhat peculiar behaviour of this function, the present method of using the exact form for $U_3(x)$ in closed form is obviously more reliable.

3. The behavior of S/κ for large values of the Reynolds number. For sufficiently large Reynolds numbers for which a universal equilibrium exists, the relation

$$S = \frac{3}{7} (30)^{1/2} \nu \frac{\int_0^\infty k^4 E(k)\, dk}{\left[\int_0^\infty k^2 E(k)\, dk \right]^{3/2}} \tag{11}$$

is exact; under these same conditions, one may therefore use Bass' equilibrium spectrum to evaluate the behaviour of the two integrals occurring in this equation. This is the procedure used by Lee [5] who found that, as R_λ *approaches* infinity,

$$S/\kappa \to 1.52. \tag{12}$$

In the course of the present investigation, however, it was found that when R_λ *equals* infinity, the behaviour of S/κ is singular with a value which differs from the one given by Eq. (12) and this fact, in itself, may not be without interest.

Thus, when R_λ *equals* infinity, the ratio S/κ can be determined by comparing the asymptotic form of the spectrum given by Kolmogoroff's universal equilibrium theory with the corresponding form given by Heisenberg's theory. For sufficiently small values of r, Kolmogoroff's prediction for the double correlation function (see, for example, [1, p. 85])

$$2\langle u^2\rangle[1 - f(r)] = (4/5S)^{2/3}(\epsilon r)^{2/3}, \tag{13}$$

where $\langle u^2\rangle$ is the mean square value of one component of the velocity, ϵ is the rate of viscous dissipation and $f(r)$ is the double velocity correlation coefficient usually denoted by this symbol, leads to the equilibrium spectrum

$$E(k) = \frac{55}{81}\cdot\frac{1}{(1/3)!}\left(\frac{4}{5S}\right)^{2/3}\epsilon^{2/3}k^{-5/3}, \tag{14}$$

and this is to be compared with Heisenberg's form of the equilibrium spectrum

$$E(k) = (8/9\kappa)^{2/3}\epsilon^{2/3}k^{-5/3}. \tag{15}$$

By equating the coefficient of $\epsilon^{2/3}k^{-5/3}$ in these two expressions, one obtains

$$\frac{S}{\kappa} = \frac{1}{810}\left[\frac{55}{(1/3)!}\right]^{3/2} \doteqdot 0.60 \tag{16}$$

and this then is the value of S/κ when R_λ *equals* infinity.

From a physical point of view, the correct limit is of course the one given by Eq. (12), and when this value is used in conjunction with the experimentally determined value of S for large values of the Reynolds number, about 0.30 [1, Fig. 6.3], we obtain the value $\kappa \doteqdot 0.20$.

4. Conclusions. The implication of this discussion then is that the ratio S/κ varies between 0.78 and 1.52 as the Reynolds number varies from zero to infinity and, from the nature of the theory, it is reasonable to suppose this variation to be monotonic. Furthermore, when the experimentally observed variation of S with the Reynolds number is taken into account, it is found that κ then varies between about 0.62 and 0.20, which neatly bracket the value (0.45 ± 0.05) suggested by Proudman [6] from his study of the correlation functions. It thus appears that the variation of κ with the Reynolds number is relatively small and cannot be said to constitute a serious criticism of Heisenberg's transfer theory.

Acknowledgement. I am indebted to Dr. G. K. Batchelor for his helpful discussion of the present work.

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NOTE ON LINEAR PROGRAMMING*

By CARL E. PEARSON (*Harvard University*)

Statement of theorem. Consider a set of m linear equations in n unknowns x_i ;

$$a_{\alpha i}x_i = b_\alpha , \tag{1}$$

where Greek indices range from 1 to m and Latin from 1 to n. The usual summation convention on repeated indices is used; $a_{\alpha i}$ and b_α are constants. Linear programming is a method of obtaining a solution (if it exists) of Eq. (1) satisfying in addition the requirements

$$x_i \geq 0, \qquad \text{all } i, \tag{2}$$

$$c_i x_i = \text{minimum}, \tag{3}$$

where the c_i are constants. The fundamental theorem used in the simplex method of Dantzig (1) is that if one solution exists, then an equivalent solution can be found in which not more than m of the x_i are non-zero; further, those columns of the matrix $(a_{\alpha i})$ which correspond to such non-zero (x_i) will be linearly independent. The usual proof of this theorem (e.g. Ref. (2)) involves tedious geometrical considerations in n-dimensional space; it therefore seems worth-while to point out that a simple direct proof exists.

Proof of theorem. Suppose (x_i') satisfies conditions (1), (2), (3). Some—perhaps all—of these (x_i') will be non-zero; say for example that x_2' , x_3' , x_6' , are alone not zero. If firstly the corresponding columns $(a_{\alpha 2}$, $a_{\alpha 3}$, $a_{\alpha 6})$ were linearly dependent, then a set of three constants K_2 , K_3 , K_6 (not all zero) would exist such that

$$A(K_2 a_{\alpha 2} + K_3 a_{\alpha 3} + K_6 a_{\alpha 6}) = 0, \qquad \text{all } \alpha \tag{4}$$

for any arbitrary constant A. Then because of Eq. (4), the new set (x_i'') defined by

$$\begin{aligned} x_i'' &= x_i' - AK_i \qquad && \text{for } i = 2, 3, 6, \\ x_i'' &= 0 && \text{for other } i \end{aligned} \tag{5}$$

satisfies Eq. (1) and, for sufficiently small A, also Eq. (2). Clearly however $c_i x_i'' < c_i x_i'$ for appropriate A, unless

$$c_2K_2 + c_3K_3 + c_6K_6 = 0. \tag{6}$$

*Received June 4, 1955.

Consequently, either the hypothesis of linear dependency has led to a contradiction so that the three columns are linearly independent, or alternatively Eq. (6) must hold. But if Eq. (6) holds, then

$$c_i x_i'' = c_i x_i' \qquad \text{for all } A,$$

and A can be chosen so that at least one of x_2'', x_3'', x_6'' vanishes, so that an equivalent solution involving fewer columns has been obtained; the same analysis may now be applied to the new solution.

It follows that, eventually, an equally good solution utilizing only linearly independent columns will be obtained. Since not more than m columns can be linearly independent, the theorem is proved.

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HEAT FLOW IN A HALF SPACE*

By WALTER P. REID† (*U. S. Naval Ordnance Test Station, China Lake, California*)

This heat flow problem has a more difficult boundary condition than usual. A formal solution is obtained by the combined use of Laplace and Fourier sine transforms.

Let the region $x > 0$ have an initial temperature which is a function of x only, $f(x)$. Assume that there is radiation of heat from the surface $x = 0$ to a finite slab, and from the other surface of the slab to surroundings whose temperature is a prescribed function of the time, $g(t)$. The slab is assumed to be thin and of high thermal conductivity so that its temperature may be considered to be uniform throughout at any time. The heat exchange by radiation between adjacent surfaces will be taken to be proportional to the difference in temperatures of the two surfaces. Mathematically, then, the problem may be stated as follows:

$$\frac{\partial u(x, t)}{\partial t} = \kappa \frac{\partial^2 u(x, t)}{\partial x^2} \qquad \text{for} \qquad x > 0,\ t > 0, \tag{1}$$

$$u(x, 0) = f(x), \tag{2}$$

$$\frac{\partial u(0, t)}{\partial x} = a[u(0, t) - v(t)], \qquad t > 0 \tag{3}$$

$$h\,\frac{\partial v(t)}{\partial t} = a[u(0, t) - v(t)] - b[v(t) - g(t)], \qquad t > 0 \tag{4}$$

$$v(0) = V. \tag{5}$$

*Received June 13, 1955.

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In these equations $u(x, t)$ is the temperature at some point in the region $x > 0$, while $v(t)$ is the temperature of the slab. The quantities a, b, h and κ are assumed to be positive constants.

In the following both the Laplace transform and the Fourier sine transform will be used. This brings up the question of notation. A bar above the letter is sometimes used to represent a transform, or a change from capital letter to lower case, or a change in letter altogether. These methods are all satisfactory if just a single transform is used, but have disadvantages in the present case. In Eq. (16), for example, notation for three different transforms is needed. In this paper a transform will be indicated simply by changing the letter used in the argument of the function, as follows:

$$u(x, s) \equiv \int_0^\infty e^{-st} u(x, t)\, dt, \tag{6}$$

$$u(\xi, t) \equiv \int_0^\infty u(x, t) \sin \xi x\, dx. \tag{7}$$

The Laplace transforms of Eqs. (1), (3) and (4) are:

$$su(x, s) - f(x) = \kappa \frac{\partial^2 u(x, s)}{\partial x^2}, \tag{8}$$

$$\frac{\partial u(0, s)}{\partial x} = a[u(0, s) - v(s)], \tag{9}$$

$$hsv(s) - hV = a[u(0, s) - v(s)] - b[v(s) - g(s)]. \tag{10}$$

Hence

$$[ab - (a + b)\partial/\partial x + ah\kappa\partial^2/\partial x^2 - h\kappa\partial^3/\partial x^3]u(0, s) = abg(s) - ahf(0) + hf'(0) + ahV. \tag{11}$$

Let

$$w(x, t) = [ab - (a + b)\partial/\partial x + ah\kappa\partial^2/\partial x^2 - h\kappa\partial^3/\partial x^3]u(x, t). \tag{12}$$

Then

$$\frac{\partial w(x, t)}{\partial t} = \kappa \frac{\partial^2 w(x, t)}{\partial x^2}, \tag{13}$$

$$sw(x, s) - w(x, 0) = \kappa \frac{\partial^2 w(x, s)}{\partial x^2}, \tag{14}$$

$$sw(\xi, s) - w(\xi, 0) = \kappa\xi w(0, s) - \xi^2 \kappa w(\xi, s), \tag{15}$$

$$w(\xi, s) = \frac{w(\xi, 0) + \kappa\xi w(0, s)}{s + \kappa\xi^2}. \tag{16}$$

From Eq. (11):

$$w(0, s) = abg(s) - ahf(0) + hf'(0) + ahV. \tag{17}$$

Take the Fourier sine transform of Eq. (12) and then eliminate the derivatives of u by integrating by parts. This gives

$$w(\xi, t) = \int_0^\infty u(x, t)[a(b - h\kappa\xi^2) \sin \xi x + \xi(a + b - h\kappa\xi^2) \cos \xi x]\, dx + ah\kappa\xi u(0, t) - h\kappa\xi \frac{\partial u(0, t)}{\partial x} \cdot \tag{18}$$

Let

$$\phi(\xi) = \int_0^\infty f(x)[a(b - h\kappa\xi^2) \sin \xi x + \xi(a + b - h\kappa\xi^2) \cos \xi x]\, dx. \tag{19}$$

Then

$$w(\xi, 0) = \phi(\xi) + ah\kappa\xi f(0) - h\kappa\xi f'(0). \tag{20}$$

So

$$w(\xi, s) = \frac{\phi(\xi) + \kappa\xi abg(s) + \kappa\xi ahV}{s + \kappa\xi^2} \cdot \tag{21}$$

Hence

$$w(\xi, t) = [\phi(\xi) + \kappa\xi ahV] \exp(-\kappa\xi^2 t) + \kappa\xi ab \int_0^t g(\tau) \exp[-\kappa\xi^2(t - \tau)]\, d\tau. \tag{22}$$

But

$$w(x, t) = \frac{2}{\pi} \int_0^\infty w(\xi, t) \sin \xi x\, d\xi. \tag{23}$$

Therefore, from Eqs. (12) and (23),

$$u(x, t) = \frac{2}{\pi} \int_0^\infty w(\xi, t) \frac{a(b - h\kappa\xi^2) \sin \xi x + \xi(a + b - h\kappa\xi^2) \cos \xi x}{a^2(b - h\kappa\xi^2)^2 + \xi^2(a + b - h\kappa\xi^2)^2}\, d\xi, \tag{24}$$

where $w(\xi, t)$ is given by Eq. (22), and $\phi(\xi)$ by Eq. (19). From Eq. (3) and (24) one may obtain $v(t)$.

NOTE ON THE SYMMETRICAL PROPERTY OF THE THERMAL CONDUCTIVITY TENSOR*

By M. LESSEN (*University of Pennsylvania*)

According to the Onsager reciprocal hypothesis, the thermal conductivity tensor among other similar properties of matter is symmetrical. In the following note it is demonstrated that for some cases of interest, the property of symmetry according to the Onsager hypothesis is not a necessary piece of information.

Assuming a space filled with a continuous medium at temperature T, having a

*Received June 15, 1955.

thermal conductivity of K_{ij} such that the conducted flow of heat through an elemental area ds having associated direction cosines l_i is

$$-(K_{ij}T_{,j})l_i\, ds.$$

The net rate of heat flow into a volume bounded by a closed surface "s" then is

$$\int_s (K_{ij}T_{,j})l_i\, ds.$$

The surface integral may be transformed to a volume integral,

$$\int_s (K_{ij}T_{,j})l_i\, ds = \int_v (K_{ij}T_{,j})_{,i}\, dv.$$

It is therefore necessary to investigate the properties of the form

$$(K_{ij}T_{,j})_{,i}\ .$$

Considering the case where

$$K_{ij} = K_{ij}(T),$$

$$(K_{ij}T_{,j})_{,i} = K_{ij}T_{,ji} + \frac{dK_{ij}}{dT} T_{,i}T_{,j}$$

$T_{,ji}$ and $T_{,i}T_{,j}$ are both symmetrical in i and j, therefore only the symmetrical portion of K_{ij} will matter in $(K_{ij}T_{,j})_{,i}$.

Since the case of $K_{ij} = K_{ij}(T)$ applies to a large class of practical applications, it is important to note, that for this case the anti-symmetrical portion of K_{ij} if it existed at all would not contribute to a first law of thermodynamic energy accounting.

A CONVERSE TO THE VIRTUAL WORK THEOREM FOR DEFORMABLE SOLIDS*

By W. S. DORN (*Aircraft Gas Turbine Development Department, General Electric Company*)

AND

A. SCHILD (*Carnegie Institute of Technology*)

1. Introduction. Consider a continuous body occupying a volume V and bounded by a closed surface S.[1] Any system of stresses σ_{ij} ,[2] satisfying the equilibrium conditions for zero body forces

$$\sigma_{ij,j} = 0, \tag{1}$$

$$\sigma_{ij} = \sigma_{ji}\ , \tag{2}$$

*Received July 27, 1955. A. Schild's participation was supported by a research grant from the National Science Foundation.

[1]It is assumed that the body is simply connected and that the surface S is composed of a finite number of pieces of each possessing a continuously turning tangent plane. All of the functions will be assumed to possess as many continuous derivatives in V and on S as are necessary for the theorems which will be used later.

[2]The subscripts range over the values 1, 2, 3 and repeated subscripts will be summed over the entire range. Subscripts following a comma denote partial differentiation with respect to Cartesian coordinates x_i , e.g., $\sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j$.

everywhere in V, and a system of virtual displacements u_i in V, must together satisfy the equation of virtual work

$$\int \sigma_{ij}\epsilon_{ij}\, dV = \int u_i\sigma_{ij}n_j\, dS, \tag{3}$$

where n_j is the unit outward normal to the surface S, and where ϵ_{ij} are the strains derived from the displacements u_i by

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{4}$$

In this note the following converse to the theorem of virtual work is proved:

If, for a symmetric tensor ϵ_{ij} given in V and for a vector u_i given on S, the virtual work equation (3) is satisfied for *all* equilibrium stresses σ_{ij} (i.e., for all σ_{ij} satisfying Eqs. (1) and (2)), *then* the ϵ_{ij} are compatible strains and are derivable, as in Eq. (4), from displacements u_i which on S have the given boundary values.

This theorem is similar to earlier results of R. V. Southwell[3], and to some recent results of H. L. Langhaar and M. Stippes.[4] In contrast to the papers quoted, however, no stress-strain relations are assumed here and our theorem is not limited to linear elasticity theory.

2. Stress functions. For each value of $i = 1, 2, 3$, Eq. (1) states that the divergence of a vector $a_j = \sigma_{ij}$ vanishes. Thus the vector may be expressed as the curl of another vector, so that

$$\sigma_{ij} = A_{ijn,n}\ , \tag{5}$$

$$A_{ijn} = -A_{inj}\ . \tag{6}$$

Then Eq. (1) is identically satisfied. From Eq. (2) we have

$$(A_{ijn} - A_{jin})_{,n} = 0.$$

By the same argument as above, it follows that

$$A_{ijn} - A_{jin} = B_{ijnm,m}\ , \tag{7}$$

$$B_{ijnm} = -B_{ijmn} = -B_{jinm}\ . \tag{8}$$

Using the symmetries given by Eqs. (6) and (8), Eq. (7) may be solved for A_{ijn}, giving

$$A_{ijn} = \tfrac{1}{2}(B_{nijm} + B_{jnmi} + B_{ijnm})_{,m}\ .$$

Therefore, by Eq. (5),

$$\sigma_{ij} = \tfrac{1}{2}(B_{nijm} + B_{jmni})_{,mn}\ . \tag{9}$$

We now define

$$P_{nijm} = \tfrac{1}{2}(B_{nijm} + B_{jmni}). \tag{10}$$

Then

$$\sigma_{ij} = P_{nijm,mn}\ , \tag{11}$$

[3] Proc. Roy. Soc. **A 154**, 4-21 (1936); *Stephen Timoshenko 60th anniversary volume*, The MacMillan Co., 1938, p. 211.

[4] J. Franklin Inst. **258**, 371-382 (1954).

where, by Eqs. (8) and (10), P_{nijm} has the symmetries

$$P_{nijm} = -P_{injm} = -P_{nimj} = P_{jmni} . \tag{12}$$

Thus any set of equilibrium stresses σ_{ij} are derivable, as in Eq. (11), from stress functions P_{nijm} which satisfy the symmetry conditions (12).

The six independent components of P_{nijm} can be expressed in terms of a symmetric second order tensor[5] T_{rs} which is a dual tensor of P_{nijm} :

$$T_{rs} = \tfrac{1}{4}\epsilon_{irn}\epsilon_{jsm}P_{nijm} , \qquad P_{nijm} = \epsilon_{irn}\epsilon_{jsm}T_{rs} , \tag{13}$$

where ϵ_{irn} is the usual completely skew-symmetric pseudotensor ($\epsilon_{123} = 1$). The six components of P_{nijm} or T_{rs} can also be identified with the stress functions χ_1 , χ_2 , χ_3 of Maxwell and ψ_1 , ψ_2 , ψ_3 of Morera as follows:[6]

$$\begin{aligned} \chi_1 &= -P_{23\,23} = T_{11} , & \psi_1 &= 2P_{31\,12} = -2T_{23} , \\ \chi_2 &= -P_{31\,31} = T_{22} , & \psi_2 &= 2P_{12\,23} = -2T_{31} , \\ \chi_3 &= -P_{12\,12} = T_{33} , & \psi_3 &= 2P_{23\,31} = -2T_{12} . \end{aligned} \tag{14}$$

For plane stress, the only non-vanishing stress function, $\chi_3 = -P_{12\,12} = T_{33}$, reduces to Airy's stress function.

3. Compatibility equations. We shall now prove the first part of the theorem stated at the end of the Introduction, i.e., that the given stresses ϵ_{ij} are compatible.

Expressing the equilibrium stresses σ_{ij} in terms of the stress functions P_{nijm} , Eq. (3) becomes

$$\int P_{nijm,mn}\epsilon_{ij}\, dV = \int u_i P_{nijm,mn} n_j\, dS. \tag{15}$$

Applying Green's theorem twice to the volume integral, we obtain

$$\int P_{nijm}\epsilon_{ij,nm}\, dV = \int [u_i P_{nijm,mn} + P_{nimj}\epsilon_{im,n} - P_{jinm,m}\epsilon_{in}]n_j\, dS. \tag{16}$$

This equation must be valid for any equilibrium stresses, and thus for an arbitrary choice of the stress functions P_{nijm} . Let P_{nijm} vanish identically outside of a small volume surrounding an interior point P of V, and let P_{nijm} be essentially constant inside the small volume. Then the surface integral on the right side of Eq. (16) vanishes. Since the P_{nijm} at P are still arbitrary except for the symmetry conditions (12), it follows that

$$\epsilon_{ij,mn} - \epsilon_{nj,im} - \epsilon_{im,jn} + \epsilon_{nm,ij} = 0 \tag{17}$$

at any point P in V. By continuity, these equations are also valid on S.

The equations (17) are the compatibility equations for strains and are necessary and sufficient conditions for the existence of a set of displacements U_i in V such that

$$\epsilon_{ij} = \tfrac{1}{2}(U_{i,j} + U_{j,i}). \tag{18}$$

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[5]C. Weber, Z. f. Ang. Math. und Mech. **28**, 193-197 (1948).

[6]A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Dover Publications, 1944, p. 88.

It remains to prove that these displacements may be so chosen, that they take on the given boundary values on S. This will be done in the next section.

4. Converse to the virtual work theorem. From Eq. (18), the (direct) theorem of virtual work for any equilibrium stresses follows by Green's theorem:

$$\int \sigma_{ij}\epsilon_{ij}\, dV = \int U_i \sigma_{ij} n_j\, dS. \tag{19}$$

Comparing with Eq. (3), we obtain

$$\int (u_i - U_i)T_i\, dS = 0, \qquad T_i \equiv \sigma_{ij}n_j\, . \tag{20}$$

This equation must hold for any choice of tractions T_i on S which are obtainable from a distribution of equilibrium stresses σ_{ij} in V. It is well known that any distribution of surface tractions T_i which is in static equilibrium can be so obtained.

Let Q and Q' be any two points on S. Choose T_i to be identically zero outside of two small areas dS^Q and $dS^{Q'}$ which respectively surround the points Q and Q'. Inside these areas, let the tractions be essentially constant and such that the forces

$$F_i^Q = T_i^Q\, dS^Q, \qquad F_i^{Q'} = T_i^{Q'}\, dS^{Q'}$$

are equal and opposite and act in the direction of the line joining QQ':

$$F_i^Q = -F_i^{Q'} = \lambda(x_i^Q - x_i^{Q'}), \qquad (\lambda \neq 0). \tag{21}$$

This system of tractions is in static equilibrium. Equation (20) now gives

$$[(u_i - U_i)_Q - (u_i - U_i)_{Q'}](x_i^Q - x_i^{Q'}) = 0. \tag{22}$$

This equation is equivalent to the geometric statement that the (infinitesimal) displacement $u_i - U_i$ leaves unchanged the distance between Q and Q'. Since this result holds for any two points on the closed surface S, it follows that $u_i - U_i$ must be a rigid body displacement:

$$u_i = U_i + \omega_{ij}x_j + \omega_i\, , \tag{23}$$

$$\omega_{ij} = -\omega_{ji} = \text{const.}, \qquad \omega_i = \text{const.}$$

Equation (23) holds on the surface S where the u_i are given. We now use Eq. (23) to define throughout the volume V a displacement u_i . It is then clear that this displacement takes on the assigned boundary values on S. From Eqs. (18) and (23) we also have

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{24}$$

This completes the proof of the converse to the virtual work theorem.

5. Conclusion. The converse to the virtual work theorem might be used in some problems on deformable solids, with given surface displacements u_i , to find among the members of an N parameter family $\epsilon_{ij}(x_1, x_2, x_3; \alpha_1, \alpha_2, \cdots, \alpha_N)$ of strain functions the one which is "least incompatible".

One could choose a set of M equilibrium stresses $\sigma_{ij}^1, \cdots, \sigma_{ij}^M$ and determine the

parameters α from the minimum principle

$$\mathrm{Min}_{\alpha_1,\cdots\alpha_N} \sum_{A=1}^{M} \left[\int \sigma_{ij}^{A}\epsilon_{ij}\, dV - \int u_i \sigma_{ij}^{A} n_j\, dS \right]^2. \tag{25}$$

Alternatively, the parameters α could be determined from a minimax principle, such as

$$\mathrm{Min}_{\alpha_1,\cdots\alpha_N} \left\{ \mathrm{Max}_{\beta_1,\cdots\beta_M} \left[\int \left(\sum_{A=1}^{M} \beta_A \sigma_{ij}^{A} \right) \epsilon_{ij}\, dV - \int u_i \left(\sum_{A=1}^{M} \beta_A \sigma_{ij}^{A} \right) n_j\, dS \right]^2 \right\}, \tag{26}$$

where the parameters β must satisfy $\sum_{A=1}^{M} \beta_A^2 = 1$.

A NOTE ON LAMINAR AXIALLY SYMMETRIC JETS*

BY MARK BERAN (*Wellesley, Mass.*)

Summary. It is shown that there is no stream function of the form $\psi = rf(\theta)$, that is compatible with the complete Navier-Stokes equations, which represents a jet issuing from a small circular hole in an axially symmetric cone.

The asymptotic velocity field of a laminar viscous jet is generally accepted to have a stream function of the form $\psi = rf(\theta)$, corresponding to self-similar flow (Schlichting [1], Squire [2], and Yatseev [3]). The authors referred to have based their discussion on the fact that this assumption of self-similarity is compatible with both the boundary layer equations, and with the full Navier-Stokes equations.

The purpose of this note is to establish a serious shortcoming of such models. It is shown that there is no continuously differentiable velocity field associated with a stream function of the form $\psi = rf(\theta)$, which satisfies the Navier-Stokes equations and also adheres to a conical wall $\theta = \alpha > 0$.

Specifically, if $\psi = rf(\theta)$, then the velocity components in the r and θ directions are respectively [4]

$$u_r = \left[\frac{1}{r \sin \theta} \right] \frac{df}{d\theta}, \tag{1}$$

$$u_\theta = \left[\frac{-1}{r \sin \theta} \right] f. \tag{2}$$

The Navier-Stokes equations are equivalent to [5]

$$f^2 = 4\nu \cos \theta f - 2\nu \sin \theta \frac{df}{d\theta} - 2(c_1 \cos^2 \theta + c_2 \cos \theta + c_3) \tag{3}$$

for suitable constants c_1, c_2, c_3. We shall show that there is no solution of (3) which (i) makes u_r and u_θ continuous for $r > 0$, and (ii) satisfies $u_r(\alpha) = u_\theta(\alpha) = 0$, for $0 < \alpha \leq \pi$.

To show this, we also consider the differentiated form of (3), which is

$$\frac{-f}{\sin 0} \frac{df}{d\theta} = 2f - 2 \sin \theta \frac{d}{d\theta} \left[\frac{1}{\sin \theta} \frac{df}{d\theta} \right] - (2c_1 \cos \theta + c_2). \tag{4}$$

*Received June 28, 1955; revised manuscript received October 5, 1955. The work on this paper was partly supported by Contract N5ori-07634 with the Office of Naval Research.

The boundary conditions implied by conditions (*i*) and (*ii*) are:

For Eq. (4): (a) $u_\theta = 0$, u_r finite, when $\theta = 0$.

(b) $u_r = u_\theta = 0$, when $\theta = \alpha$.

For Eq. (5): (c) $u_\theta = 0$, u_r finite, $\partial u_r/\partial\theta$ finite, when $\theta = 0$.*

These yield respectively the equations:

$$c_1 + c_2 + c_3 = 0, \tag{5}$$

$$c_1 \cos^2 \alpha + c_2 \cos \alpha + c_3 = 0, \tag{6}$$

$$2c_1 + c_2 = 0. \tag{7}$$

These have $c_1 = c_2 = c_3 = 0$ as their only solution.

As Squire [5] has shown, the general solution of Eq. (4) with $c_1 = c_2 = c_3 = 0$ is:

$$f = \frac{2\nu \sin^2 \theta}{a + 1 - \cos \theta}, \tag{8}$$

where a is an arbitrary constant.

Referring again to the boundary condition $u_r(\alpha) = u_\theta(\alpha) = 0$, we see that there is no finite value of $a(a = \infty$ yields a satisfactory but trivial solution) which satisfies this boundary condition, no matter what value of α is chosen.

Thus we have shown that there is no non-trivial solution of the form $\psi = rf(\theta)$ that is compatible with the Navier-Stokes equations and the boundary conditions (*i*) and (*ii*).

The author wishes to thank Prof. Garrett Birkhoff for suggesting the problem and for his helpful advice.

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[5] H. B. Squire, Quart. J. Mech. Appl. Math. **4**, 321–29 (1950)

*It is assumed that when these boundary conditions are substituted in Eqs. (4) and (5) that the limit $\theta \to 0$ is taken, since $\theta = 0$ is a singular point in the spherical polar coordinate system.

HEAT CONDUCTION IN SEMI-INFINITE SOLID IN CONTACT WITH LINEARLY INCREASING MASS OF FLUID*

By C. C. CHAO and J. H. WEINER (*Columbia University*)

Introduction. Problems of transient heat conduction in which the surface of a solid is in contact with a well-stirred fluid have been the subject of numerous investigations.[1] In all previous cases studied, the mass of the fluid has been considered constant. However, it is sometimes of interest to know the temperature in the solid and fluid

*Received August 9, 1955.

[1]A review of previous work is found in [1], pp. 16-17.

while the latter is being poured onto the solid. This is so, for example, in the analysis of the filling stage of a casting process.

In this paper, the problem of a semi-infinite solid in perfect contact with a well-stirred fluid whose mass is initially zero and increases linearly with time is considered. The principal boundary condition involved has variable coefficients and the solution of the problem by the Laplace transform technique requires the solution of a differential equation with the Laplace transform parameter as independent variable. Numerical results are presented.

Problem. The problem described above is formulated mathematically as follows. (Property values are taken independent of temperature; the initial temperature of the solid is taken as zero.)

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}; \qquad x > 0, \qquad t > 0, \tag{1}$$

$$\lim_{t \to 0} u(x, t) = 0; \qquad x > 0, \tag{2}$$

$$\lim_{x \to \infty} u(x, t) = 0; \qquad t \geq 0, \tag{3}$$

$$K \frac{\partial u}{\partial x} = mc\left[\frac{\partial}{\partial t}(tu) - V\right]; \quad \text{at} \quad x = 0, \qquad t > 0, \tag{4}$$

where $u(x, t)$ is the temperature in the solid at distance x from the surface at time t after the start of pour, K is the thermal conductivity of the solid, κ its thermal diffusivity, V is the pouring temperature of the liquid, c is the specific heat of the liquid, and m is its (constant) mass rate of flow.

Equation (4) is based on the assumptions that the fluid is well-stirred and is in perfect contact with the solid, that is, that the entire mass of liquid is at the temperature of the solid surface. It is derived from a heat balance on the liquid.

Solution. The solution of the problem is obtained by use of the Laplace transform. Let $U(x, p) = \int_0^\infty \epsilon^{-pt} u(x, t)\, dt$ denote the Laplace transform of $u(x, t)$. Then the Laplace transforms of Equations (1)-(4) are:

$$\frac{\partial^2 U}{\partial x^2} = \frac{pU}{\kappa}; \qquad x > 0, \tag{5}$$

$$\lim_{x \to \infty} U(x, p) = 0, \tag{6}$$

$$K \frac{\partial U}{\partial x} = -mc\left[p \frac{\partial U}{\partial p} + \frac{V}{p}\right]; \qquad x = 0. \tag{7}$$

The transform of Eq. (4), Eq. (7), is obtained by using the operational properties of the Laplace transform ([2], operations 3 and 8, p. 294).

The solution of Eq. (5) which satisfies Eq. (6) is:

$$U = A(p) \exp\left[-(p/\kappa)^{1/2} x\right]. \tag{8}$$

Substitution of Eq. (8) into Eq. (7) leads to the following differential equation for $A(p)$:

$$\frac{dA}{dp} - \frac{sA}{p^{1/2}} = \frac{V}{p^2}, \tag{9}$$

where $s = K/mc\kappa^{1/2}$.

The solution of Eq. (9) is

$$A(p) = \frac{V}{p} - 4s^2V[(2sp^{1/2})^{-1} + \exp(2sp^{1/2})Ei(-2sp^{1/2})] + C\exp(2sp^{1/2}), \tag{10}$$

where $Ei(-x) = \int_x^\infty y^{-1}e^{-y}\,dy$ is the exponential integral. It is seen that the arbitrary constant, C, must be zero if $A(p)$ is to have a sectionally continuous inverse transform of exponential order.

Therefore, substitution of Eq. (10) with $C = 0$ in Eq. (8) yields the transform of $u(x, t)$ as

$$U(x, p) = V\{p^{-1} - 4s^2[(2sp^{1/2})^{-1} + \exp(2sp^{1/2})Ei(-2sp^{1/2})]\}\exp[-(p/\kappa)^{1/2}x]. \tag{11}$$

The following transform pair is known ([4], p. 268, pair 31):

$$L^{-1}[(2s)^{-1} + p\exp(2sp)Ei(-2sp)] = (t + 2s)^{-2}.$$

Therefore,

$$\begin{aligned} L^{-1}\{[(2s)^{-1} + p\exp(2sp)Ei(-2sp)]\exp(-xp\kappa^{-1/2})\} &= (t - x\kappa^{-1/2} + 2s)^{-2}, \quad t > x\kappa^{-1/2}. \\ &= 0, \qquad\qquad t < x\kappa^{-1/2}. \end{aligned}$$

The following property of Laplace transforms is known ([1], p. 243): If

$$L^{-1}[W(p)] = w(t),$$

then

$$L^{-1}[W(p^{1/2})p^{-1/2}] = (\pi t)^{-1/2}\int_0^\infty w(y)\exp(-y^2/4t)\,dy$$

therefore,

$$\begin{aligned} &L^{-1}[\exp(2sp^{1/2})Ei(-2sp^{1/2}) + (2sp^{1/2})^{-1}]\exp[-(p/\kappa)^{1/2}x] \\ &\qquad = (\pi t)^{-1/2}\int_{x/\kappa^{1/2}}^\infty (y - x\kappa^{-1/2} + 2s)^{-2}\exp(-y^2/4t)\,dy, \\ &\qquad = (2t\pi^{1/2})^{-1}\int_{x/2(\kappa t)^{1/2}}^\infty \{\xi - [x/2(\kappa t)^{1/2}] + st^{-1/2}\}^{-2}\exp(-\xi^2)\,d\xi. \end{aligned}$$

Use of the above transformation pair in Eq. (11) together with the known pair ([1], p. 381, pair 8)

$$L^{-1}\{p^{-1}\exp[-(p/\kappa)^{1/2}x]\} = \mathrm{erfc}\,[x/2(\kappa t)^{1/2}]$$

in Eq. (11) yields

$$u(x, t) = V\Big[\operatorname{erfc}\,[x/2(\kappa t)^{1/2}] - 2s^2t^{-1}\pi^{-1/2}\int_{x/2(\kappa t)^{1/2}}^{\infty}\{\xi - [x/2(\kappa t)^{1/2}] + st^{-1/2}\}^{-2}\exp(-\xi^2)\,d\xi\Big]. \tag{12}$$

The above analysis has been purely formal. It may be verified, however, by direct substitution that the above function satisfies Eqs. (1)-(4) defining the boundary value problem. The interchange of differentiation and integration required in this verification may be readily justified.

Numerical results. The temperature of the liquid, which is equal to $u(0, t)$, is of particular interest. By use of integration by parts, it may be put in the following form:

$$u(0, t) = V\Big\{1 - 2s(\pi t)^{-1/2} + 2s^2t^{-1} - 4s^3(\pi t^3)^{-1/2}\int_0^{\infty}(\xi + st^{-1/2})^{-1}\exp(-\xi^2)\,d\xi\Big\}. \tag{13}$$

Numerical values for the latter integral have been tabulated by Goodwin and Staton [3]. However, in the evaluation of Eq. (13), it was found necessary to carry their asymptotic expansion further in order to obtain sufficiently accurate results. A graph of $u(0, t)/V$ as function of t/s^2 is shown in Fig. 1.

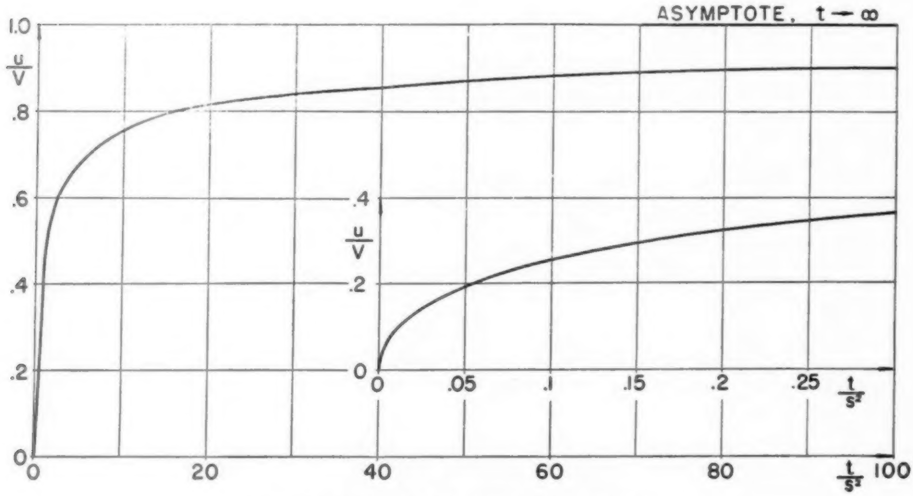

Fig. 1 Variation of fluid temperature with time.

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ERRORS IN ASYMPTOTIC SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS*

By F. W. J. OLVER (*National Physical Laboratory, Teddington, England*)

In a recent paper [1] R. L. Evans has described a general method for estimating errors in asymptotic expansions of solutions of ordinary linear differential equations, the basis of which is as follows.

Let

$$y(x) \sim \sum_{\alpha=0}^{\infty} c(\alpha)x^{\rho-\alpha} \tag{4†}$$

be the formal asymptotic expansion for large x of a solution $y(x)$ of the differential equation

$$L(y) \equiv \sum_{\nu=0}^{n} p_\nu(x)y^{(n-\nu)} = 0, \tag{3}$$

in which

$$p_\nu(x) = \sum_{\mu=-m(\nu)}^{n(\nu)} b_{\nu,\mu}x^{-\mu} \qquad [\nu = 0, 1, \cdots, n \text{ and each integer } n(\nu) \text{ is finite}].$$

The error in question is the difference between $y(x)$ and the sum of the first N terms in its asymptotic expansion, and is given by

$$v(x) = y(x) - u(x), \tag{6}$$

where

$$u(x) = \sum_{\alpha=0}^{N-1} c(\alpha)x^{\rho-\alpha}. \tag{5}$$

It satisfies a linear differential equation of the form

$$M(v) - L(v) = L(u), \tag{11}$$

where M is a linear differential operator of order n whose coefficients can be found from the $p_\nu(x)$, and whose term in the nth derivative cancels in (11) with that of $L(v)$. Accordingly (11) is of order $n - 1$. In particular, if $n = 2$ equation (11) is of the first order and yields immediately an indefinite integral for $v(x)$ from which bounds for $v(x)$ can be obtained.

The purpose of this note is to point out that Evans' result is incorrect. No non-trivial differential equation of the form (11) can exist. If it did, the relation $y(x) = v(x) + u(x)$, obtained from (6), would enable us to assert that the general solution of any nth order differential equation of the form (3) can always be made to depend on the solution of a non-homogeneous differential equation of order $n - 1$. In particular, we would always be able to express the general solution of any second-order equation of the form (3) as a finite combination of indefinite integrals of elementary functions.

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*Received August 16, 1955.

†The notation and equation numbers are the same as in [1].

This conclusion is borne out by an example given in [1] in which the method is applied to the equation

$$y'' + \left(-2 + \frac{1}{x}\right)y' - \left(\frac{1}{x} + \frac{n^2}{x^2}\right)y = 0,$$

with solutions $e^x I_n(x)$, $e^x K_n(x)$. It leads to the result

$$y = \sum_{\alpha=0}^{N-1} c(\alpha)x^{-\frac{1}{2}-\alpha} + (\text{constant}) \times x^{-N/2}\, e^x \int_x^\infty \xi^{-(N+1)/2} e^{-\xi}\, d\xi,$$

where the constant depends on N but not on x. This suggests that the derivatives of $x^{N/2}I_n(x)$ and $x^{N/2}K_n(x)$ can be expressed as finite combinations of elementary functions, which is incorrect.

If the analysis given in [1] is examined, it is found that the relations given between Eqs. (9) and (10) do not follow from (9) and (6). It would appear that the correct forms of (9) and these relations are given by

$$\sigma = \rho - N, \qquad v(x) \sim \sum_{\alpha=0}^{\infty} C(\alpha)x^{\sigma-\alpha},$$

in which event (7) transforms into itself with c, ρ replaced by C, σ respectively; in particular the equation between (10) and (10a) is to be replaced by

$$\begin{aligned} &\{A_1(\sigma_a - \alpha - 2) + A_3\}C_a(\alpha + 2) + A_5 C_a(\alpha) \\ &\quad + \{(\sigma_a - \alpha - 1)(\sigma_a - \alpha - 2) + A_2(\sigma_a - \alpha - 1) + A_4\}C_a(\alpha + 1) = 0. \qquad \text{(A)} \end{aligned}$$

In constructing the differential equation for $v(x)$ which "corresponds" to (A) it must be remembered that $C(-1), C(-2), \cdots, C(-N)$ are non-zero, accordingly this equation is non-homogeneous. Its correct form is in fact given by

$$L(v) = -L(u),$$

a result which is otherwise immediately obtainable from (3) and (6). The proposed scheme is accordingly nugatory.

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Reference

1. R. L. Evans, *Errors in asymptotic solutions of linear ordinary differential equations*, Quart. Appl. Math. 12, 295 (1954)

BOOK REVIEWS

(Continued from p. 170)

Tables of integral transforms. Based, in part, on notes left by Harry Bateman. Compiled by Staff of the Bateman Manuscript Project. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1954. Volume I, xx + 391 pp., $7.50. Volume II, xvi + 451 pp., $8.00.

The two volumes in question arise from the Bateman Manuscript Project at the California Institute of Technology under the support of the Office of Naval Research. Prepared in part from notes left by the late Harry Bateman by A. Erdelyi, W. Magnus, F. Oberbettinger, and F. G. Tricomi, these two volumes contain transform tables arranged as follows: *Volume I.* Fourier Sine, Cosine and Exponential Transforms 118 pp., Laplace Transforms 173 pp., Mellin Transforms 59 pp. *Volume II.* Bessel Transforms. Hankel Transforms of the form $\int_0^\infty f(x)J_\nu(xy)(xy)^{\frac{1}{2}}\,dx$, 83 pp., and the same for $1/\nu(xy)$, $K_\nu(x, y)$ and $H_\nu(x, y)$, 75 pp., Kontrovich-Lebedev Transforms, 3 pp. Miscellaneous Transforms. Fractional Integral, Stieltjes and Hilbert Transforms, 78 pp. Integrals of Higher Transcendental Functions. Orthogonal Polynomials, Gamma functions and related functions, Legendre functions, Bessel functions and Hypergeometric functions, 152 pp.

Tables are very well arranged. The notation is explained in appendix to each volume. The printing job is really excellent.

ROHN TRUELL

Proceedings of the International Conference of Theoretical Physics. Kyoto and Tokyo, September, 1953. The Organizing Committee, International Conference of Theoretical Physics, Science Council of Japan, Ueno Park, Tokyo, 1954. xxviii + 942 pp. $10.00. (postage $1.00 extra).

This book of more than nine hundred pages is a detailed account of a conference held in Japan in September 1952 which covered almost all of the currently interesting topics of theoretical physics. No attempt will be made to mention all of the authors who contributed to the conference. In fact it is unfortunately impossible even to list the titles of all of the papers. There were about 125 papers presented. Because all of the topics covered were represented by the recognized leaders in these fields the relative emphasis in various fields is indicated below by the number of papers presented and the number of pages taken up in the proceedings. The main topics were: I. *Field Theory and Elementary Particles.* Papers and discussion concerned with Field Theory, Cosmic Rays and V-particles, Pions, Coupling Theory, and Nuclear Forces occupy approximately one third of this book. Forty five papers were presented in this area of physics with the result that these proceedings contain probably the best summary available of recent work. II. *Nuclear Physics.* Nuclear Reactions. Shell Structure and Beta Decay were the subject matter of a dozen papers and eighty pages of these proceedings. III. *Statistical Mechanics.* The Theory of Polymers, Liquids, Transport Phenomena, Irreversible Processes and General Methods of Statistical Mechanics were topics covered in twenty one papers and 180 pages of the proceedings.

IV. *Molecules and Solids.* Dislocations, Molecular Theory, Metals, Electron Theory of Intrinsic Magnetization, Antiferro- and Ferrimagnetism, Magnetic Resonance, Dielectrics, and color centers were the subjects in that section of the conference dealing with solid state theory. Thirty seven papers cover 311 pages of the proceedings.

V. *Liquid Helium and Superconductivity.* Theories of Liquid Helium and Superconductivity are discussed in ten papers covering about sixty pages of the proceedings.—This conference and the proceedings appear to have made a large contribution by putting into print an outline of the present state of theoretical physics as it is seen by a large number of physicists from all countries.

ROHN TRUELL

Introduction to theoretical mechanics. By Robert A. Becker. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1954. xiii + 420 pp. $8.00.

This is a text book on a level appropriate for junior or senior year courses. It is based on a course for students in engineering physics, and the choice of topics, examples, and problems has a flavor of physics rather than of engineering. Some 80 worked and 400 unworked problems are given. Vectors are used throughout, but tensor or matrix concepts are not mentioned. The chapters on plane and general motion of a rigid body seem especially well written and complete, considering the elementary level of mathematics used. The chapter on vibrating systems, however, seems inadequate since it is confined mainly to properties of normal coordinates, and the standard techniques of combining normal modes are not introduced. The principle of virtual displacements appears not to be given the importance it deserves as an analytical tool in both statics and dynamics (in deriving Lagrange's equations, for example). Apart from these minor criticisms the reviewer was well impressed by this as a carefully written and reliable basic text.

P. S. Symonds

Geometrical mechanics and de Broglie waves. By J. L. Synge. Cambridge University Press, 1954. vi + 167 pp. $4.75.

This monograph presents a rather unique amalgamation of several disciplines from the author's unusual collection of interests. The result of this synthesis is an approach to relativistic particle dynamics based upon Hamilton's methods of geometrical optics, the Minkowski geometry of special relativity and the particle-wave concept of de Broglie.

After a short introductory chapter, the author devotes about half of the book to a generalization of the Hamilton methods to the four-dimensional Minkowski space-time. Although the methods are described as Hamilton's methods of optics, it is pointed out that these differ only formally from Hamilton's analysis of classical dynamics. The former, however, is in a form more easily adapted to the generalization in question. In this generalization, the rays of optics (trajectories of dynamics) become time-like world lines of particles. The waves of optics (2-dimensional surfaces) become 3-waves in the Minkowski space describing the history of a 2-wave.

One of the very interesting results of this procedure is that two velocities, a ray velocity and a wave velocity emerge from the theory in a natural way. These are interpreted to correspond respectively to the particle and wave velocities of de Broglie waves. These two velocities are obtained, however, solely from geometrical considerations without any discussion of wave interference or other concepts customarily involved in the association of particle velocity with a group velocity.

Not until chapter IV does the author attempt to assign a phase to these 3-waves. He does so then only for the purpose of incorporating into the theory a "primitive quantization" in much the way that one obtains physical optics from geometrical optics by the assignment of a phase to the wave front. Several special applications of this relativistic quantum mechanics are considered, including the hydrogen atom quantization, the Zeeman effect and interference of a particle from two holes.

The final chapter is concerned with the further generalization of the methods to the two body problem in an 8-dimensional product of two Minkowski spaces.

The author points out that his primary concern is to develop a "coherent mathematical theory"; that physical interpretation is considered of secondary importance. Indeed the theory is not all inclusive from a physicist's point of view, for there is no obvious way of incorporating into the theory the concepts of spin, or "second quantization" and even the "first quantization" according to this scheme has been tested on only a few problems. Where the theory is known to give correct or good approximate results, however, this approach to the physical problem furnishes a very elegant geometrical interpretation.

The purpose of the theory is well stated, the postulates are well defined and the development proceeds in a systematic fashion. Although this reviewer thinks that the book gives an excellent presentation of the ideas that it attempts to convey, he fears that it will not receive the circulation it deserves because it does not seem to belong specifically to any of the many narrow channels of current scientific interest. The physical principles underlying the theory have been known since 1928. If this book could have been published then, it would undoubtedly have been an immediate sensation.

Gordon F. Newell

Modern developments in fluid dynamics. Two Volumes. Edited by L. Howarth. Oxford University Press, London, 1953. xvi + 875 pp. $17.00 per set.

These two volumes were developed as companion volumes to "*Modern developments in fluid dynamics*" edited by S. Goldstein, which appeared in 1938. These latter books have become indispensable to all those working in fluid dynamics, since they bring together in a concise and accurate form the accumulated understanding of fluid mechanics phenomena as of the time of their publication. The present companion volumes on high speed flow are also sponsored by the fluid motion panel and attempt to similarly summarize developments in this latter field.

The publication of two new volumes instead of a revision of the older volumes is possible because much of the new work in fluid mechanics which grew out of wartime problems has been in the field of high velocities. However, any publication in this latter field must be carried out under the disadvantage of attempting to describe the important phenomena without violation of security restrictions which arose during the war and which have only been partially relaxed. As in the previous volumes, a group of distinguished authors have contributed sections to the new work.

Roughly, the first volume is devoted to discussions of theory and the second volume to questions of experimental results. The first two chapters are devoted to a general introduction to the various high speed phenomena and to the equations in both vector and tensor notation required for their description. In these chapters, as throughout the book, the editor has attempted to make clear the distinction between flows in which the entropy is constant for each particle, so that its change of state is isentropic, and flows in which the entropy is constant everywhere. These latter are described as homentropic. While this distinction is clearly a significant one, the reviewer was sometimes confused by the words used and was left with the feeling that the change was somewhat of a fetish.

Chapter 3 on the characteristics method is followed by Chapter 4 in which shock waves and their formation and various interactions are described. After thus treating fundamentals in some detail, Chapter 5 presents the few available exact solutions, vortex flow, radial flow, spiral flow and Prandtl-Meyer flow including their various applications.

Chapter 6 and 7 treat the one-dimensional approximation with applications to nozzles and jets, and the various techniques available by use of the hodograph plane. Then comes several chapters dealing with approximate methods. The various linearization procedures and numerous of their airfoil, planform, tube, and jet applications are followed by available solutions to unsteady problems, primarily unsteady airfoil characteristics.

Chapter 10, the last one in the first volume, treats the general theory of boundary layers including skin friction, aerodynamic heating, heat transfer, stability, and boundary layer-shock wave interaction. The first volume contains 475 pages.

The second volume begins with Chapter 11 devoted to experimental methods which is broken down into four sections. The first, wind tunnels and moving bodies, discusses general tunnel characteristics and supersonic nozzle design together with very brief statements about intermittent tunnels, shock tubes, and moving body tests. It appears to the reviewer that the latter two methods have been relatively neglected since they receive less than 2 pages of attention.

The second section devoted to uncertainties and corrections arising in tunnel tests is followed by a section on measurements and their reliability, and finally, a section on visualization in which each of the common optical systems is treated in some detail. In Chapter 12, the experimental results available on flow past airfoils and cylinders are reported together with some comparisons with theoretical predictions. In this chapter, the lift, drag moment and their various derivatives are discussed with respect to variation with Mach number, Reynolds number, angle of attack, angle of sweep, section parameters and controls.

Chapter 13, devoted to flow past bodies of revolution is devoted primarily to experimental and theoretical results on projectiles. The base pressure receives very considerable consideration. The range testing technique and the theory of the spinning shell are discussed. Chapters 11, 12, 13, which constitute three-quarters of the second volume are the chapters wherein restrictions on publication make a complete discussion of the problems difficult. The authors have attempted an important piece of writing using generally available information to bring out as clearly as possible the essential features of experimental results in the high-speed aerodynamic field.

The final Chapter 14 on heat transfer would better be considered a revision of what appears as the last chapter with the same title in the earlier volumes. Many sub-sections are identical to the previous volume, although many others have been rather completely rewritten, so as to bring them up to date.

In this heat transfer chapter, there is no attempt to limit the discussions to "high speed flow" although this phase of the subject has similarly been revised from the previous editions.

These two volumes are, the reviewer believes, destined to take their position beside the earlier two volumes as the major publications in the field of fluid mechanics during the middle of our century.

HOWARD W. EMMONS

Vector and tensor analysis. By G. E. Hay. Dover Publications, Inc., New York, 1953. viii + 193 pp. $2.75.

Many excellent books are available on both the vector and tensor calculus and their applications. The present book contains no preface, and accordingly its purpose must be inferred from its contents. The book seeks to develop the essential parts of the vector and tensor calculus together with some discussion of geometrical and physical applications. This, a formidable task to perform within a monograph, is made more difficult by reducing to a minimum the supposed background knowledge of the reader. The greater part (156 pages) of the book is devoted to the vector calculus, covering the usual material up to Green's and Stokes' theorems, and including applications in geometry and rigid body dynamics. This part is well written, and a clear account is given. The remainder (37 pages) of the book is devoted to the tensor calculus with very brief mention of its applications. Here the coverage (this includes weighted tensors and covariant differentiation) is achieved only through extreme condensation of analysis, and the average reader would find it difficult to work through the problems.

The book contains a few incorrect statements and misprints. A partial bibliography is given but it is surprising to find no reference to several important standard texts.

H. G. HOPKINS

Proceedings of the Eastern Joint Computer Conference, held by the Joint Computer Committee of the Association for Computing Machinery, American Institute of Electrical Engineers, and the Institute of Radio Engineers, Inc., in Philadelphia, December 8–10, 1954. American Institute of Electrical Engineers, New York, 1955. 92 pp. $3.00.

The theme of this conference was the design and application of small digital computers. The Proceedings Volume commences with an introductory talk by C. W. Adams, who defines small digital computers by means of examples. He identifies two classes: (1) Those built up from punch card equipment, which he calls "big-little computers", and (2) Those built up around a magnetic drum, which he calls "little-big computers". (The significance of these names for the two classes would perhaps be clearer if the hyphen in each case were moved forward by a word.) Computers of the second class typically have an internal storage capacity of between 10,000 and 40,000 decimal digits.

The papers presented at such a conference fall into several recognizable categories. First, there is the category of survey papers, which discuss available equipment or current trends in development. In the present case, this category is represented by a good survey of currently available small digital computers presented by A. J. Perlis. Second, there are the papers devoted to engineering design problems. For various reasons, some of which are quite obvious, these have a tendency to be somewhat vague when presented by engineers working for private industry. The volume under review contains three or four papers of this type covering the following subjects: pulse packing density, magnetic core circuits, and automation of information retrieval.

A third recognizable type of paper is the company report which describes a specific proprietary piece of equipment. As is usual at computer conferences held by the engineering societies, this category is well represented in the current volume.

Finally, there are the contributions of those interested in the applications. In the present conference the emphasis here was mainly on business problems. However, there are two papers in the Proceedings Volume with some mathematical content. The first is a paper by H. M. Gurk and Morris Rubinoff on the numerical solution of differential equations. This paper contains some interesting stability charts for the

finite difference analogue of the equation $\dot{x} = \lambda \dot{x}$ where λ is complex. The second mathematical paper is a simplified discussion of the use of digital computers in optical ray tracing, by N. A. Finkelstein.

The reviewer found this volume to be a relatively interesting one of its kind and the editorial work was excellent. But this fact did not deflect the reviewer from the pursuit of a project which he has had in mind for some time. The project is to found a Society for the Prevention of the Publication of Proceedings of Symposia. The holding of symposia is a fine thing indeed, but the morning after (which sometimes takes place two years later) is a different matter altogether. The papers with substantial content presented at a symposium should be, and almost always are, published elsewhere than in the Proceedings, with fuller detail. The papers which do not have substantial content should not be published at all.

Membership in the new Society is free. Any member who can prove that he has been instrumental in the suppression of a symposium proceedings volume will be made a Fellow.

J. H. Curtiss

Table of Everett's interpolation coefficients. By E. W. Dijkstra and A. Van Wijngaarden. Excelsior's Photo-Offset, The Hague, 1955. $1.58.

This table was constructed on the electronic computer ARRA of the Mathematisch Centrum in Amsterdam on request of Empresa Nacional Bazan de Construcciones Navales Militares in Madrid. It gives to seven decimal places the values of the coefficients of the second, fourth, and sixth central differences in Everett's interpolation formula.

This table is especially convenient for three reasons; a) Entries for the argument p are given at intervals of 0.0001 so that interpolation is rarely needed in the table itself. b) All the coefficients needed for any given interpolation are printed in the single line of the table, i.e., the coefficients for p and for $1 - p$ are both given on the same line. c) The range extends from $p = 0$ to $p = 1$, so that it is not necessary to change the procedure for $p > 1/2$. As a result of b) and c) all entries appear twice in the table, once at p and once at $1 - p$.

The use of sixth order differences in Everett's formula is equivalent to using eight equally spaced values of the function for the interpolation, or in other words, using a polynomial approximation of seventh degree.

W. E. Milne

Tables of the cumulative binomial probability distribution. By The Staff of the Computation Laboratory. Harvard University Press, Cambridge, 1955. lxi + 503 pp. $8.00.

This, the first set of tables to be computed on the new Harvard Mark IV Calculator, gives values of the cumulative binomial probability distribution

$$E(n, r, p) = \sum_{x=r}^{n} p^x (1 - p)^{n-x} \, n!/x!(n - x)!$$

to five decimal places for integer values of r and n. The unique feature of the tables is that n ranges in varying steps from 1 to 1000 as compared with a previous high of 150. Since only integer r and n are included, the tables are not quite as complete for small n as the old tables by Pearson for the incomplete beta function which included half integer values as well. The incomplete beta function differs from $E(n, r, p)$ only in a trivial way.

The introduction includes seventeen illustrative applications in addition to a discussion of the analytic properties of the function.

G. F. Newell

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